

STATIONARY SPHERICALLY - SYMMETRIC CHARGE STRUCTURES

APPENDIX A

ANALYTICAL SOLUTION OF THE ELECTRODYNAMIC EQUATIONS

A.1 CONFIGURATION, POSTULATES, NOTATION

A.1.0 Configuration and Problem Studied:

Consider a spherically symmetric, compressible, nonviscous charged 'fluid', defined by a mass density $\rho(r, t)$ and a charge density $q(r, t)$. The fluid is in radial motion with velocity field $v(r, t)$ in an electric field $E(r, t)$ caused by and acting on its charge distribution, producing an acceleration field $a(r, t)$. Find the motion of the charge configuration in response to its own field, and a consistent set of solutions for $E(r, t)$, $v(r, t)$, $a(r, t)$, $q(r, t)$, and $\rho(r, t)$ describing the motion.

Abbreviated notation for functions used:

$$E = E(r, t), q = q(r, t), \rho = \rho(r, t), v = v(r, t), a = a(r, t), R = R(r), T = T(t)$$

A.1.1 Definition of Charge Density:

Maxwell's equation relating electric field \mathbf{E} to charge density q :

$$q = \epsilon \operatorname{div} \mathbf{E}$$

i.e. in spherical symmetry:

$$q = \epsilon \left(2 \frac{E}{r} + \left(\frac{\partial}{\partial r} E \right) \right), \quad (\text{A-1})$$

where ϵ is the permittivity (assumed to be constant in space and time).

A.1.2 Definition of Current Density:

Maxwell's equation relating magnetic field \mathbf{B} to current density $q \mathbf{v}$ and displacement current

$$\epsilon c^2 \operatorname{curl} \mathbf{B} = q \mathbf{v} + \epsilon \left(\frac{\partial}{\partial t} \mathbf{E} \right)$$

i.e. in spherical symmetry (where $\operatorname{curl} \mathbf{B} = 0$):

$$q \mathbf{v} + \epsilon \left(\frac{\partial}{\partial t} \mathbf{E} \right) = 0, \quad (\text{A-2})$$

where \mathbf{v} is the velocity at radius r and time t (Eulerian picture).

A.1.3 Conservation of Charge:

Continuity equation for charge density $q(r,t)$ (redundant from A.1.1 and A.1.2, cf Sect A.2.3):

$$\left(\frac{\partial}{\partial t} q\right) + \text{div}(q \mathbf{v}) = 0$$

i e in spherical symmetry:

$$\left(\frac{\partial}{\partial t} q\right) + \frac{\frac{\partial}{\partial r} r^2 q v}{r^2} = 0, \quad (\text{A-3})$$

A.1.4 Conservation of Mass:

Continuity equation for inertial mass density ρ , equal to energy density/ c^2 :

$$\left(\frac{\partial}{\partial t} \rho\right) + \text{div}(\rho \mathbf{v}) = 0$$

i e in spherical symmetry:

$$\left(\frac{\partial}{\partial t} \rho\right) + \frac{\frac{\partial}{\partial r} r^2 \rho v}{r^2} = 0, \quad (\text{A-4})$$

where ρ designates the total inertial mass (cf Sect A.2.4).

A.1.5 Definition of Electric Force:

Newton's second law connecting time derivative of momentum to electric field (conservation of inertial mass according to Eq(A-4) implicitly assumed when evaluating time derivative of momentum, cf Sect A.2.5):

$$\frac{d \mathbf{v}}{dt} = \frac{q \mathbf{E}}{\rho}$$

i e in spherical symmetry and with 'convective derivative' (cf Jackson p 471, Feynmann p II-40-4) of velocity $v(r,t)$:

$$\left(\frac{\partial}{\partial t} v\right) + v \left(\frac{\partial}{\partial r} v\right) = \frac{q E}{\rho}, \quad (A-5)$$

A.1.6 Space-Time Separability:

The electric field $E(r,t)$ is assumed to be separable in one factor $R(r)$ which is a function of radius r alone, and one factor $T(t)$ which is a function of time t alone (cf discussion in Sect A.2.6):

$$E = R T, \quad (A-6)$$

A.1.7 Notations

This Appendix (except Section 2) is a computer print-out of calculations from symbolic mathematics programs written in the Maple V.2 language (Char et al, MapleV)

Notation for constants of integration introduced by Maple: $_C1, _C2$

Constants introduced or redefined by the author: $A_, B_, C_, C3_, \eta, \sigma$.

Maple denotes substitutions it makes in formulas as: $\%1, \%2$ etc.

A further discussion of the above postulates will be found in Sect A.2. The derivation of the electrodynamic equations continues in Sect A.3.

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A.2 COMMENTS TO THE POSTULATES

A.2.1 Comments on Spherical Symmetry

Since there is no magnetic field in spherical symmetry and thus no Poynting vector, a hypothetical quasi-stationary, spherically symmetric charge configuration will not lose energy by electromagnetic radiation. On the other hand, it will be expected that all modes initially present other than spherically symmetric ones will quickly dissipate through such radiation, so that after an initial transient phase only the spherically symmetric modes remain.

A.2.2 Comments on Maxwell's Equations

The other two Maxwell equations:

$$\mathit{div} \mathbf{B} = 0;$$

and

$$\mathit{curl} \mathbf{E} = - \partial \mathbf{B} / \partial t$$

are trivially valid in spherical symmetry. Implicit use of them by solving the generalized wave equation in \mathbf{E} (A Bergström, Phys Rev D8, p 4394) instead of the approach below will result in an extra constant of integration.

A.2.3 Comments on Redundancy in the Postulates

Under the assumptions made in Sect A.1, there is a certain redundancy in the Postulates. Eq(A-3) can be derived from Eq(A-1) and Eq(A-2) by taking the divergence of Eq(A-2) and using Eq(A-1) to eliminate E . This redundancy is due to

the assumption made in Sect A.1.1 that the (vacuum) permittivity ϵ is constant, i.e. independent of strength of the electric field $E(r,t)$.

A.2.4 Comments on the Inertial Mass Density

The inertial mass density $\rho(r,t)$ used in the present study is the total inertial mass density, which takes into account the masses of the individual charges, their effective inertial mass due to their interactions with neighboring charges, the inertia contribution due to the electrostatic field which the charges produce, possible relativistic momentum increase, etc. Although thus dependent not only on the individual masses but also on the charge configuration as a whole, the total inertial mass density ρ nevertheless obeys the continuity equation Eq(A-4). Actually, Eqs (A-4) and (A-5) are two components of the equation of motion, $\nabla \cdot \mathbf{T} = 0$, for the stress-energy tensor \mathbf{T} describing conservation of 4-momentum for the system in differential form, see Misner, Thorne & Wheeler, *Gravitation*, § 5.10, where this is discussed in detail.

A.2.5 Comments on the Electric Volume Force

Assume a fluid with mass density $\rho = \rho(\mathbf{r},t)$ and charge density $q = q(\mathbf{r},t)$ in motion with velocity field $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$ in an electric field $\mathbf{E} = \mathbf{E}(\mathbf{r},t)$. Conservation of momentum for a volume element with (infinitesimal) volume V , i.e. with mass $M = \rho V$ and charge $Q = q V = q M / \rho$, gives:

$$d(M \mathbf{v})/dt = Q \mathbf{E} .$$

We also assume the mass density ρ to obey the continuity equation in Sect A.1.3:

$$\partial\rho/\partial t + \nabla\cdot(\rho \mathbf{v}) = 0 . \quad (\text{A-3})$$

It is instructive to derive the electric volume force for two different types of volume elements, as will be done below. (In both cases we assume the electric permittivity ϵ to be a constant, otherwise extra, dielectric forces will also be involved).

A.2.5.1 Case 1: Volume element with constant mass

Follow the motion of a volume element enclosing constant mass M , i e

$$d(M \mathbf{v})/dt = q M \mathbf{E} / \rho ,$$

or

$$M dv/dt = q M \mathbf{E} / \rho ,$$

i e

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla) \mathbf{v} = q \mathbf{E} / \rho . \quad (\text{A-5a})$$

This is the equation of motion given in the literature (cf, e g, J D Jackson, Classical Electrodynamics, p 471).

In spherical symmetry we have in particular:

$$\partial v/\partial t + v \partial v/\partial r = q E / \rho , \quad (\text{A-5})$$

which is the equation given in Sect A.1.5.

A.2.5.2 Case 2: Volume element with constant volume

Follow the motion of a volume element enclosing constant volume V (and disregarding for the moment matter flowing in or out through the boundary of the volume element). We then have

$$d(\rho V \mathbf{v})/dt = q V \mathbf{E} ,$$

or

$$V d(\rho \mathbf{v})/dt = q V \mathbf{E} ,$$

i e

$$\partial(\rho \mathbf{v})/\partial t + (\mathbf{v} \cdot \nabla)(\rho \mathbf{v}) = q \mathbf{E} .$$

In addition to the two terms on the left hand side of the above equation we must in this case also take into account the increase or decrease of momentum due to matter flowing in or out through the boundary of the infinitesimal volume element, which from Gauss' theorem is equal to $\rho \mathbf{v} (\nabla \cdot \mathbf{v})$ per unit volume. The complete equation of motion for the volume element is thus

$$\partial(\rho \mathbf{v})/\partial t + (\mathbf{v} \cdot \nabla)(\rho \mathbf{v}) + \rho \mathbf{v} (\nabla \cdot \mathbf{v}) = q \mathbf{E} . \quad (\text{A-5b})$$

The expression $(\mathbf{v} \cdot \nabla)(\rho \mathbf{v})$ can be evaluated by using the identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} .$$

With $\mathbf{a} = \mathbf{v}$ and $\mathbf{b} = \rho \mathbf{v}$ (i e $\mathbf{a} \times \mathbf{b} = 0$) we then have

$$(\mathbf{v} \cdot \nabla)(\rho \mathbf{v}) = \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) - \rho \mathbf{v} (\nabla \cdot \mathbf{v}) .$$

Inserting the above equation into Eq(A-5b) we get after expanding the time derivative and dividing by ρ

$$\partial \mathbf{v} / \partial t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} [\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v})] / \rho - \mathbf{v} (\nabla \cdot \mathbf{v}) + \mathbf{v} (\nabla \cdot \mathbf{v}) = q \mathbf{E} / \rho .$$

The third term vanishes according to the continuity equation Eq(A-3), and we get

$$\partial \mathbf{v} / \partial t + (\mathbf{v} \cdot \nabla) \mathbf{v} = q \mathbf{E} / \rho ,$$

in agreement with Eq (A-5a) above.

A.2.6 Comments on Space-Time Separability

Due to the nonlinearity of the equations, Postulate A.1.6 of separability of space and time dependence imposes strong restrictions on possible solutions, and - as we shall see - in fact limits possible solutions to one class of functions only. It might be argued that albeit there are stationary solutions in this class of functions, maybe other solutions with mixed space and time dependence coexist with these stationary solutions and provide modes of motion by which the stationary solutions will decay.

However, even though the equations governing the field form a nonlinear system, this system can be linearized in every small surrounding around a particular value (except at possible singularities), and the solution is then there unique. Thus, even though there may exist expanding modes with mixed space and time dependence the charge configuration will, due to this local uniqueness of the solution, be expected to remain in a quasi-stationary mode once it has got into such a mode.

A related question is then how to create this stationary structure in the first place, which must obviously be through some process channel which is no longer available in the stationary state, otherwise the structure could decay through the same channel through which it was formed. One method to create a self-confined charge structure as discussed here, and avoiding the problem of it decaying by the same route, is by forming an intense, initially cylindrical electrical discharge, which is caused to pinch due to the magnetic field of the discharge current. The pinched discharge is then allowed to deform by its inherent instabilities, preferentially by 'sausage'-type instabilities, leaving - after the current is switched off - a highly energetic, essentially spherically-symmetric charge configuration with a radial structure as derived below, and which corresponds to a stationary charge structure.

A.3 DERIVATION OF PARTIAL DIFFERENTIAL EQUATION FOR $E(r,t) = R(r) T(t)$

Insert assumed factorization of $E(r,t)$ from Eq(A-6) into the Maxwell equation Eq(A-1) to get charge density $q(r,t)$ expressed in $R(r)$ and $T(t)$:

$$q = \epsilon \left(2 \frac{R T}{r} + \left(\frac{\partial}{\partial r} R T \right) \right)$$

$$q = \frac{\epsilon T \left(2 R + \left(\frac{\partial}{\partial r} R \right) r \right)}{r}, \quad (A-7)$$

Insert charge density $q(r,t)$ from Eq(A-7) into the Maxwell equation Eq(A-2) and solve velocity $v(r,t)$ expressed in $R(r)$ and $T(t)$:

$$\frac{\epsilon T \left(2 R + \left(\frac{\partial}{\partial r} R \right) r \right) v}{r} + \epsilon \left(\frac{\partial}{\partial t} R T \right) = 0$$

ie

$$v = - \frac{R \left(\frac{\partial}{\partial t} T \right) r}{T \left(2 R + \left(\frac{\partial}{\partial r} R \right) r \right)}, \quad (A-8)$$

Insert velocity $v(r,t)$ from Eq(A-8) into Newton's second law, Eq(A-5), to get a partial differential equation in $R(r)$ and $T(t)$:

$$\left(\frac{\partial}{\partial t} - \frac{R \left(\frac{\partial}{\partial t} T \right) r}{T \% 1} \right) - \frac{R \left(\frac{\partial}{\partial t} T \right) r \left(\frac{\partial}{\partial r} - \frac{R \left(\frac{\partial}{\partial t} T \right) r}{T \% 1} \right)}{T \% 1} = \frac{\epsilon T^2 \% 1 R}{r \rho}$$

$$\% 1 := 2 R + \left(\frac{\partial}{\partial r} R \right) r$$

Evaluate and collect terms in powers of T :

$$-\frac{\left(\frac{\partial^2}{\partial t^2} T\right) r}{\left(2R + \left(\frac{\partial}{\partial r} R\right) r\right) T^2} + \frac{\left(\frac{\partial}{\partial t} T\right)^2 r \left(6R^2 + 4\left(\frac{\partial}{\partial r} R\right) r R + 2\left(\frac{\partial}{\partial r} R\right)^2 r^2 - R r^2 \left(\frac{\partial^2}{\partial r^2} R\right)\right)}{\left(2R + \left(\frac{\partial}{\partial r} R\right) r\right)^3 T^3} =$$

$$\frac{T \varepsilon \left(2R + \left(\frac{\partial}{\partial r} R\right) r\right)}{r \rho}$$

or after some further simplification:

$$-\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{\left(\frac{\partial}{\partial t} T\right)^2 \left(6R^2 + 4\left(\frac{\partial}{\partial r} R\right) r R + 2\left(\frac{\partial}{\partial r} R\right)^2 r^2 - R r^2 \left(\frac{\partial^2}{\partial r^2} R\right)\right)}{\left(2R + \left(\frac{\partial}{\partial r} R\right) r\right)^2 T^2} =$$

$$\frac{T^2 \left(2R + \left(\frac{\partial}{\partial r} R\right) r\right)^2 \varepsilon}{r^2 \rho}, \quad (\text{A-9})$$

A.3.1 Comments on Two Special Cases

One case in which Eq(A-9) is consistent with the separability assumption, Eq(A-6), is the special case when the second term on the left hand side in Eq(A-9) and the term on the right hand side degenerate into a single term, in which case we thus have an equation of the form:

$$\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{(\mu - 1) \left(\frac{\partial}{\partial t} T\right)^2}{T^2} = 0, \quad (\text{A-10})$$

which has the solution:

$$T = (t C1 + C2) \left(\frac{1}{\mu} \right), \quad (A-11)$$

Closer analysis of Eq(A-9) in this degenerate case reveals the further requirement $\mu = -1$. In the present context of stationary solutions we do not consider the above solution, and study only the general case when the terms do not degenerate.

As a second special case, we study the ansatz:

$$R = \frac{A}{r^n}, \quad (A-12)$$

in which case Eq(A-9) becomes:

$$-\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{(n-3) \left(\frac{\partial}{\partial t} T \right)^2}{(-2+n) T^2} = \frac{A^2 (-2+n)^2 \epsilon T^2}{(r^n)^2 r^2 \rho}, \quad (A-13)$$

In the asymptotic case of a Coulomb field ($n=2$), the second term on the left hand side in Eq(A-13) dominates and we have the solution:

$$R = \frac{A}{r^2}, \quad (A-14)$$

$$T = C1, \quad (A-15)$$

The causal mechanism behind this static nature of the Coulomb field will be understood from the general solution of Eq(A-9) which is given in the following.

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A.4 GENERAL SOLUTION OF RADIAL FACTOR $R(r)$

In order for the separability assumption, Eq(A-6), to be generally valid we must require that the terms in Eq(A-9) are functions of t only, i.e. that the r -dependent coefficients equate to constants. From the second term on the left hand side we then get the condition (σ is a constant):

$$\frac{6R^2 + 4\left(\frac{\partial}{\partial r}R\right)rR + 2\left(\frac{\partial}{\partial r}R\right)^2r^2 - Rr^2\left(\frac{\partial^2}{\partial r^2}R\right)}{\left(2R + \left(\frac{\partial}{\partial r}R\right)r\right)^2} = \sigma, \quad (A-16)$$

Eq(A-16) can be solved by the substitution:

$$S = S(r)$$

$$R = e^{\int S dr + C_2}, \quad (A-17)$$

From Eq(A-17) we have:

$$\frac{\partial}{\partial r}R = SR, \quad (A-18)$$

$$\frac{\partial^2}{\partial r^2}R = \left(\frac{\partial}{\partial r}S\right)R + S^2R, \quad (A-19)$$

which transform Eq(A-16) into the equation:

$$\frac{6 + 4Sr + S^2r^2 - r^2\left(\frac{\partial}{\partial r}S\right)}{(2 + Sr)^2} = \sigma$$

with the following general solution for S :

$$S = - \frac{3r - 2\sigma r - 2_{-C1}}{r^2 - \sigma r^2 - _{C1}r}$$

and with the corresponding solution for R from Eq(A-17):

$$R = e^{\int - \frac{3r - 2\sigma r - 2_{-C1}}{r^2 - \sigma r^2 - _{C1}r} dr + _{C2}}$$

or

$$R = \frac{(r\sigma - r + _{C1}) \left(\frac{1}{\sigma - 1} \right) e^{-C2}}{r^2}$$

We can rewrite this analytical solution to Eq(A-16) as (A_{-} denotes a scaling constant):

$$\sigma = 1 - \frac{1}{\eta}, \quad (A-20)$$

$$R = \frac{(r - _{C1}\eta)^{(-\eta)} \eta^\eta A_{-}}{r^2}, \quad (A-21)$$

(Later in Eq(A-29) it will be shown that the constant of integration $_{C1}=0$).

In the following sections various properties and corollaries will be derived from the radial solution in Eq(A-21), as far as can be done without detailed knowledge of the time dependent factor $T(t)$ which will be available only after the solution later in Sect A.5 of the time dependent equation.

A.4.1 Corollary: Derivation of Charge Density as Function of r and $T(t)$

Inserting the radial solution from Eq(A-21) into Eq(A-7) we obtain the following expression for the charge density $q(r,t)$:

$$q = \frac{\epsilon T \left(2 \frac{(r - CI \eta)^{(-\eta)} \eta \eta A_-}{r^2} + \left(\frac{\partial}{\partial r} \frac{(r - CI \eta)^{(-\eta)} \eta \eta A_-}{r^2} \right) r \right)}{r}$$

or evaluated:

$$q = - \frac{\epsilon T A_- (r - CI \eta)^{(-\eta - 1)} \eta (\eta + 1)}{r^2}, \quad (A-22)$$

A.4.2 Corollary: Derivation of Velocity as Function of r and $T(t)$

Inserting the radial solution from Eq(A-21) into Eq(A-8) we obtain the following expression for the velocity $v(r,t)$ of the charged medium:

$$v = - \frac{(r - CI \eta)^{(-\eta)} \eta \eta A_- \left(\frac{\partial}{\partial t} T \right)}{r T \left(2 \frac{(r - CI \eta)^{(-\eta)} \eta \eta A_-}{r^2} + \left(\frac{\partial}{\partial r} \frac{(r - CI \eta)^{(-\eta)} \eta \eta A_-}{r^2} \right) r \right)}$$

or simplified:

$$v = \frac{(r - CI \eta) \left(\frac{\partial}{\partial t} T \right)}{\eta T}, \quad (A-23)$$

A.4.3 Check Consistency with Continuity Equation for Charge Density, Eq(A-3)

Insert charge density $q(r,t)$ from Eq(A-22) and velocity $v(r,t)$ from Eq(A-23) into the continuity equation for charge density $q(r,t)$ in Eq(A-3):

$$\left(\frac{\partial}{\partial t} - \frac{\epsilon T A_- (r - _CI \eta)^{(-\eta - 1)} \eta^{(\eta + 1)}}{r^2} \right) + \frac{\partial}{\partial r} - \frac{\epsilon A_- (r - _CI \eta)^{(-\eta - 1)} \eta^{(\eta + 1)} (r - _CI \eta) \left(\frac{\partial}{\partial t} T \right)}{r^2} = 0$$

which simplifies to

$$0 = 0$$

Eq(A-22) and Eq(A-23) are thus consistent with the continuity equation for charge density.

A.4.4 Derivation of Mass Density as Function of r and $T(t)$

Inserting the radial solution from Eq(A-21) into Eq(A-9) we obtain:

$$-\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{\left(\frac{\partial}{\partial t} T \right)^2}{T^2} - \frac{\left(\frac{\partial}{\partial t} T \right)^2}{\eta T^2} = \frac{T^2 (r - _CI \eta)^{(-2 \eta - 2)} \eta^{(2 \eta + 2)} A_-^2 \epsilon}{r^4 \rho}, \quad (A-24)$$

Again, the right hand side of Eq(A-24) must equate to a function of t only, say $f(t)$, in order for the separability assumption, Eq(A-6), to be generally valid:

$$-\frac{\partial^2 T}{\partial t^2} + \frac{\left(\frac{\partial T}{\partial t}\right)^2}{T^2} - \frac{\left(\frac{\partial T}{\partial t}\right)^2}{\eta T^2} = f(t), \quad (\text{A-25})$$

The inertial mass density must then have the following form:

$$\rho = \frac{T^2 (r - \text{CI } \eta)^{(-2 \eta - 2)} \eta^{(2 \eta + 2)} A_{-2} \varepsilon}{f(t) r^4}, \quad (\text{A-26})$$

where the time-dependent factor $f(t)$ is so far unknown. This function has to be known also for the subsequent solution in Sect A.5 of the time dependent part $T(t)$ of the field from the differential equation Eq(A-25).

A.4.4.1 Lemma: Derivation of an Equation for the Function $f(t)$

Inserting the expression for the inertial mass density from Eq(A-26) and the expression for the velocity from Eq(A-23) into the continuity equation for the inertial mass density in Eq(A-4), we obtain:

$$2 \frac{T \%2 \%1 A_{-2} \varepsilon \left(\frac{\partial T}{\partial t}\right)}{f(t) r^4} - \frac{T^2 \%2 \%1 A_{-2} \varepsilon \left(\frac{\partial f(t)}{\partial t}\right)}{f(t)^2 r^4} + \left(\begin{aligned} & -2 \frac{T \%2 \%1 A_{-2} \varepsilon (r - \text{CI } \eta) \left(\frac{\partial T}{\partial t}\right)}{r^3 f(t) \eta} + \frac{T \%2 (-2 \eta - 2) \%1 A_{-2} \varepsilon \left(\frac{\partial T}{\partial t}\right)}{r^2 f(t) \eta} \\ & + \frac{T \%2 \%1 A_{-2} \varepsilon \left(\frac{\partial T}{\partial t}\right)}{r^2 f(t) \eta} \end{aligned} \right) / r^2 = 0$$

$$\%1 := \eta^{(2 \eta + 2)}$$

$$\%2 := (r - \text{CI } \eta)^{(-2 \eta - 2)}$$

or simplified:

$$- T(r - _CI \eta)^{(-2 \eta - 2)} \eta^{(2 \eta + 2)} A_{-2} \epsilon$$

$$\left(T \left(\frac{\partial}{\partial t} f(t) \right) r \eta + 3 \left(\frac{\partial}{\partial t} T \right) f(t) r - 2 \left(\frac{\partial}{\partial t} T \right) f(t) _CI \eta \right) / (f(t)^2 r^5 \eta) = 0$$

This condition is satisfied if:

$$\frac{\partial}{\partial t} T = \frac{T \left(\frac{\partial}{\partial t} f(t) \right) r \eta}{f(t) (-3 r + 2 _CI \eta)}, \quad (A-27)$$

Insert Eq(A-27) into Eq(A-25) to get a differential equation for $f(t)$:

$$-\frac{\left(\frac{\partial^2}{\partial t^2} f(t) \right) r \eta}{f(t) (-3 r + 2 _CI \eta)} + \frac{\left(\frac{\partial}{\partial t} f(t) \right)^2 r \eta}{f(t)^2 (-3 r + 2 _CI \eta)} - \frac{\left(\frac{\partial}{\partial t} f(t) \right)^2 r^2 \eta}{f(t)^2 (-3 r + 2 _CI \eta)^2} = f(t),$$

(A-28)

In order for $f(t)$ to be a function of t only we set for the constant of integration $_CI$:

$$_CI = 0, \quad (A-29)$$

and thus obtain the equation:

$$\frac{1}{3} \frac{\left(\frac{\partial^2}{\partial t^2} f(t) \right) \eta}{f(t)} - \frac{4}{9} \frac{\left(\frac{\partial}{\partial t} f(t) \right)^2 \eta}{f(t)^2} = f(t), \quad (A-30)$$

The analytical solution of Eq(A-30) will be discussed in Sect A.4.4.4 below. For the present purpose, however, it is sufficient to have $f(t)$ as a function of $T(t)$, which will be accomplished as follows.

A.4.4.2 Lemma: Solution of the Function $f(t)$ Expressed in $T(t)$

Make the ansatz (cf the discussion of this ansatz in Sect A.4.4.4 below):

$$f(t) = B_- T^n, \quad (A-31)$$

where B_- denotes a scaling constant.

Insert the ansatz Eq(A-31) into Eq(A-30):

$$\frac{\frac{\partial^2}{\partial t^2} T}{T} - \frac{1}{3} \frac{\left(\frac{\partial}{\partial t} T\right)^2}{T^2} (n+3) = 3 \frac{B_- T^n}{\eta n}, \quad (A-32)$$

Similarly, insert the ansatz Eq(A-31) into Eq(A-25):

$$-\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{\left(\frac{\partial}{\partial t} T\right)^2}{\eta T^2} (\eta - 1) = B_- T^n, \quad (A-33)$$

After equating T^n from Eq(A-32) and Eq(A-33) we get (apart again from a special case corresponding to Eq(A-10)):

$$\frac{1}{3} \frac{(n+3)\eta}{\eta-1} = 1, \quad (A-34)$$

$$-3 \frac{1}{\eta n} = 1, \quad (A-35)$$

Eq(A-34) and Eq(A-35) have the solution:

$$\left\{ \eta = \eta, n = -3 \frac{1}{\eta} \right\}$$

i e

$$n = -3 \frac{1}{\eta}, \quad (A-36)$$

From Eq(A-36) and Eq(A-31) we then get:

$$f(t) = B_T \left(-3 \frac{1}{\eta} \right), \quad (A-37)$$

A.4.4.3 Corollary: Mass Density as Function of r and $T(t)$

Using the results in Eq(A-29) and Eq(A-37), the mass density in Eq(A-26) becomes:

$$\rho = \frac{T^2 r^{(-2\eta - 2)} \eta^{(2\eta + 2)} A_{-2} \epsilon}{B_T \left(-3 \frac{1}{\eta} \right) r^4}, \quad (A-38)$$

A.4.4.4 Comments on the Ansatz Eq(A-31)

The ordinary differential equation Eq(A-47) in Sect A.5 below has an analytical solution as given in Eq(A-62) and Eq(A-63). Comparing the above differential equation Eq(A-30) in $f(t)$ with Eq(A-47), we conclude that Eq(A-30) is indeed a special case of Eq(A-47) and thus has a solution with a parametric representation of the form given in Eq(A-62) and Eq(A-63). This parametric solution is consistent with the ansatz Eq(A-31) and the expression given in Eq(A-37).

A.4.5 Check Consistency with Continuity Equation for Mass Density, Eq(A-4)

Eq(A-38) thus constitutes a restriction on the form of possible inertial mass densities in order for the assumption on separability Eq(A-6) to be valid. Nevertheless, we can check that this expression for the inertial mass density together with the expression for the velocity $v(r,t)$ from Eq(A-23) indeed satisfy the continuity equation for the inertial

mass density (or energy density) in Eq(A-4):

$$0 = 0$$

A.4.6 Corollary: Derivation of Acceleration as Function of r and $T(t)$

Inserting the charge density $q(r,t)$ from Eq(A-22) and the mass density $\rho(r, t)$ from Eq(A-38) into Eq(A-5), and with the electric field from Eq(A-6) and Eq(A-21), we get the following expression for the acceleration $a(r,t)$:

$$a = - \frac{r B_{-} T \left(-3 \frac{1}{\eta} \right)}{\eta}, \quad (A-39)$$

A.4.7 Corollary: Derivation of Total Charge as Function of r and $T(t)$

From Gauss' theorem and inserting Eq(A-21) (with $_{-}C1$ from Eq(A-29)) into Eq(A-6), we conclude that the time dependent total charge $Q(r,t)$ inside (a fixed) radius r is:

$$Q(r, t) = 4 \pi r^2 \epsilon E$$

$$Q(r, t) = 4 \pi \epsilon r^{(-\eta)} \eta \eta A_{-} T, \quad (A-40)$$

On the other hand, we can integrate the charge density $q(r,t)$ in Eq(A-22) to obtain the total charge Q_s of a spherical shell extending from an inner radius δ (possibly infinitesimal) to outer radius r :

$$Q_s = \int_{\delta}^r -4 \pi \epsilon T A_{-} r^{(-\eta - 1)} \eta^{(\eta + 1)} dr$$

$$Q_s = 4 \frac{\pi \varepsilon T A_- \eta^\eta}{r^\eta} - 4 \frac{\pi \varepsilon T A_- \eta^\eta}{\delta^\eta}, \quad (\text{A-41})$$

When $\eta < 0$, the second term on the right hand side of Eq(A-41) is zero for $\delta = 0$, and the charge of a sphere with radius r is then equal to that of a shell extending from a vanishing inner radius δ to outer radius r :

$$Q(r, t) = Q_s, \quad (\text{A-42})$$

When $\eta > 0$, on the other hand, the radial field derived in Eq(A-21) and the charge density derived in Eq(A-22) are compatible only if there is also a central core at or around $r=0$ with charge Q_o , where:

$$Q_o = 4 \frac{\pi \varepsilon T A_- \eta^\eta}{\delta^\eta}, \quad (\text{A-43})$$

so that the total charge $Q(r,t)$ inside radius r is equal to the sum of the charge Q_s of a spherical shell from radius δ to radius r PLUS the charge Q_o inside the central core. When $\eta > 0$, the central core does not vanish in the limit $\delta = 0$ as it does in the case $\eta < 0$.

$$Q(r, t) = Q_s + Q_o, \quad (\text{A-44})$$

Expressed in terms of the total charge $Q(r,t)$ in Eq(A-40), the scaling factor A_- in the radial part of the electric field in Eq(A-21) thus becomes:

$$A_- = \frac{1}{4} \frac{Q(r, t)}{\pi \varepsilon r^{(-\eta)} \eta^\eta T}, \quad (\text{A-45})$$

and using Eq(A-45) we can also express the charge density in Eq(A-22) in terms of the total charge $Q(r,t)$ as follows (where we have also used Eq(A-29) for $_{-}C1$):

$$q = - \frac{1}{4} \frac{Q(r, t) \eta}{\pi r^3}, \quad (\text{A-46})$$

A.5 GENERAL SOLUTION OF TIME-DEPENDENT FACTOR $T(t)$

With the result in Eq(A-36) the differential equation Eq(A-33) for $T(t)$ becomes:

$$-\frac{\partial^2 T}{\partial t^2} + \frac{\left(\frac{\partial T}{\partial t}\right)^2}{\eta T^2} = B_- T \left(-3 \frac{1}{\eta}\right), \quad (A-47)$$

Eq(A-47) has the analytical solution (C_- and $_{-}C2$ denote constants of integration):

$$t = \frac{\eta \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-}}{C_-} + \frac{\eta^2 B_- \ln \left(2 \sqrt{-C_-} \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-} - 2 C_- T \left(\frac{1}{\eta}\right) + 2 \eta B_- \right) - _{C}2}{(-C_-)^{3/2}}$$

$$t = - \frac{\eta \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-}}{C_-} - \frac{\eta^2 B_- \ln \left(2 \sqrt{-C_-} \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-} - 2 C_- T \left(\frac{1}{\eta}\right) + 2 \eta B_- \right) - _{C}2}{(-C_-)^{3/2}}$$

Disregarding for the moment the direction of time, we write for simplicity:

$$t = - \frac{\eta \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-}}{C_-} - \frac{\eta^2 B_- \ln \left(2 \sqrt{-C_-} \sqrt{2 T \left(\frac{1}{\eta}\right) \eta B_- - \left(T \left(\frac{1}{\eta}\right)\right)^2 C_-} - 2 C_- T \left(\frac{1}{\eta}\right) + 2 \eta B_- \right)}{(-C_-)^{3/2}} - C_2, \quad (A-48)$$

We can rewrite Eq(A-48) in a more transparent form by the substitution:

$$T = \left(2 \frac{\eta B_- \cos(\theta)^2}{C_-} \right)^\eta, \quad (A-49)$$

Substituting Eq(A-49) in Eq(A-48) we thus obtain:

$$t = - \left(2 i \eta^2 B_- \cos(\theta) \sqrt{-1 + \cos(\theta)^2} + i \eta^2 B_- \ln(2) + i \eta^2 B_- \ln(B_-) + i \eta^2 B_- \ln(\eta) + i \eta^2 B_- \ln \left(-2 \cos(\theta) \sqrt{-1 + \cos(\theta)^2} - 2 \cos(\theta)^2 + 1 \right) + C_2 C_-^{3/2} \right) / C_-^{3/2}, \quad (A-50)$$

Use the familiar relationship:

$$\cos(\theta)^2 = 1 - \sin(\theta)^2, \quad (A-51)$$

to rewrite Eq(A-50) as:

$$t = - \left(-2 \eta^2 B_- \cos(\theta) \sin(\theta) + i \eta^2 B_- \ln(2) + i \eta^2 B_- \ln(B_-) + i \eta^2 B_- \ln(\eta) + i \eta^2 B_- \ln \left(-2 i \cos(\theta) \sin(\theta) - 1 + 2 \sin(\theta)^2 \right) + C_2 C_-^{3/2} \right) / C_-^{3/2}, \quad (A-52)$$

Use the relationships for double angles:

$$\cos(\theta) = \frac{1}{2} \frac{\sin(2\theta)}{\sin(\theta)}, \quad (A-53)$$

$$\sin(\theta)^2 = \frac{1}{2} - \frac{1}{2} \cos(2\theta), \quad (A-54)$$

to further simplify Eq(A-52) to:

$$t = - \left(-\eta^2 B_- \sin(2\theta) + i \eta^2 B_- \ln(2) + i \eta^2 B_- \ln(B_-) + i \eta^2 B_- \ln(\eta) + i \eta^2 B_- \ln(-i \sin(2\theta) - \cos(2\theta)) + {}_C2 C_-^{3/2} \right) / C_-^{3/2}, \quad (A-55)$$

From Euler's formula we have:

$$i \sin(2\theta) + \cos(2\theta) = e^{2i\theta}$$

or

$$-i \sin(2\theta) - \cos(2\theta) = e^{2i\left(\theta + \frac{1}{2}\pi\right)}, \quad (A-56)$$

which can be used to rewrite Eq(A-55) as:

$$t = - \left(-\eta^2 B_- \sin(2\theta) + i \eta^2 B_- \ln(2) + i \eta^2 B_- \ln(B_-) + i \eta^2 B_- \ln(\eta) - 2 \eta^2 B_- \theta - \eta^2 B_- \pi + {}_C2 C_-^{3/2} \right) / C_-^{3/2}, \quad (A-57)$$

Choosing the constant of integration ${}_C2$ so that $t=0$ when $\theta=0$:

$${}_C2 = - \frac{i \eta^2 B_- \ln(2) + i \eta^2 B_- \ln(B_-) + i \eta^2 B_- \ln(\eta) - \eta^2 B_- \pi}{C_-^{3/2}}$$

we can write Eq(A-57) as:

$$t = \frac{\eta^2 B_- (\sin(2\theta) + 2\theta)}{C_-^{3/2}}, \quad (A-58)$$

Using the relation for double angles:

$$\cos(\theta) = \sqrt{\frac{1}{2} \cos(2\theta) + \frac{1}{2}}, \quad (A-59)$$

the expression for $T(t)$ in Eq(A-49) can similarly be rewritten as:

$$T = \left(\frac{\eta B_- (\cos(2\theta) + 1)}{C_-} \right)^\eta, \quad (A-60)$$

In Eq(A-58) and Eq(A-60) we further set:

$$\theta = \frac{1}{2} \tau, \quad (A-61)$$

and then obtain the following final expressions for the time t and the time dependent part of the electric field $T(t)$ as functions of the parameter τ :

$$t = \frac{\eta^2 B_- (\sin(\tau) + \tau)}{C_-^{3/2}}, \quad (A-62)$$

$$T = \left(\frac{\eta B_- (\cos(\tau) + 1)}{C_-} \right)^\eta, \quad (A-63)$$

NB The direction of time may always be chosen to be positive as discussed above in connection with the solution Eq(A-48) to the differential equation Eq(A-47) for $T(t)$.

A.5.1 Check Solution by Back Substitution

We calculate:

$$\frac{\partial}{\partial t} T = \frac{\frac{\partial}{\partial \tau} T}{\frac{\partial \tau}{\partial t}}$$

ie from Eq(A-62) and Eq(A-63) we get:

$$\frac{\partial}{\partial t} T = - \frac{\left(\frac{\eta B_- (\cos(\tau) + 1)}{C_-} \right)^\eta \sin(\tau) C_-^{3/2}}{\eta (\cos(\tau) + 1)^2 B_-}, \quad (A-64)$$

By inserting Eq(A-64) and Eq(A-63) into Eq(A-47), we can verify that the parametric solution for $T(t)$ in Eq(A-63) and Eq(A-62) indeed satisfies the differential equation Eq(A-47):

$$\frac{C_-^3}{\left(\cos(\tau)^3 + 3 \cos(\tau)^2 + 3 \cos(\tau) + 1 \right) \eta^3 B_-^2} = \frac{C_-^3}{\left(\cos(\tau)^3 + 3 \cos(\tau)^2 + 3 \cos(\tau) + 1 \right) \eta^3 B_-^2}$$

A.5.2 Corollary: Derivation of Velocity as Function of r and t

Inserting Eq(A-64) and Eq(A-63) (and Eq(A-29)) into the expression for $v(r, T(t))$ in Eq(A-23) we get the following expression for the velocity $v(r, t)$ at radius r and time t (Eulerian picture, cf Goldstein p 427, Clemmow & Dougherty p 32):

$$v = - \frac{r \sin(\tau) C_-^{3/2}}{\eta^2 (\cos(\tau) + 1)^2 B_-}, \quad (A-65)$$

A.5.3 Corollary: Derivation of Acceleration as Function of r and t

Inserting Eq(A-63) into the expression for $a(r, T(t))$ in Eq(A-39) we get the following expression for the acceleration at radius r and time t (Eulerian picture):

$$a = - \frac{r C_-^3}{\eta^4 B_-^2 (\cos(\tau) + 1)^3}, \quad (\text{A-66})$$

A.5.4 Corollary: Derivation of Position of a Volume Element as Function of t

We see from Eq(A-62) and Eq(A-65) that the charged medium is momentarily at rest at $t=0$. We assume the position of a particular volume element to be $r(0)$ at $t=0$. From Eq(A-65) follows that the motion $r(t)$ of the volume element (Lagrangian picture, c f Goldstein p 427, Clemmow & Dougherty p 32) is described by the differential equation:

$$\frac{\partial}{\partial t} r(t) = - \frac{r(t) \sin(\tau) C_-^{3/2}}{\eta^2 (\cos(\tau) + 1)^2 B_-}$$

Changing variables we have:

$$\frac{\frac{\partial}{\partial \tau} r(\tau)}{\frac{\partial}{\partial \tau} t(\tau)} = - \frac{r(\tau) \sin(\tau) C_-^{3/2}}{\eta^2 (\cos(\tau) + 1)^2 B_-}, \quad (\text{A-67})$$

The derivative of $t(\tau)$ can be calculated from Eq(A-62):

$$\frac{\partial}{\partial \tau} t(\tau) = \frac{\eta^2 B_- (\cos(\tau) + 1)}{C_-^{3/2}}, \quad (\text{A-68})$$

Inserting Eq(A-68) in Eq(A-67) we obtain the following differential equation for $r(\tau)$:

$$\frac{\left(\frac{\partial}{\partial \tau} r(\tau)\right) C_-^{3/2}}{\eta^2 B_- (\cos(\tau) + 1)} = - \frac{r(\tau) \sin(\tau) C_-^{3/2}}{\eta^2 (\cos(\tau) + 1)^2 B_-}$$

or simplified:

$$\frac{\partial}{\partial \tau} r(\tau) = - \frac{r(\tau) \sin(\tau)}{\cos(\tau) + 1}, \quad (A-69)$$

with the solution:

$$r(\tau) = C3_- (\cos(\tau) + 1), \quad (A-70)$$

With the constant of integration $C3_-$ calculated from $r(0)$, we have:

$$r(\tau) = \frac{1}{2} r(0) (\cos(\tau) + 1), \quad (A-71)$$

A.5.5 Corollary: Derivation of Acceleration of Volume Element as Function of t

By inserting the expression for the position of a volume element as function of time from Eq(A-71) into Eq(A-66), we can calculate the acceleration of a co-moving volume element of the charge configuration (Lagrangian picture):

$$a = - \frac{1}{2} \frac{r(0) C_-^3}{(\cos(\tau) + 1)^2 \eta^4 B_-^2}, \quad (A-72)$$

A.5.6 Corollary: Derivation of Velocity of Volume Element as Function of t

We now calculate the velocity of a co-moving volume element of the charge

configuration in the Lagrangian picture, assuming the initial position of the volume element at $t=0$ to be $r(0)$. As a check of the consistency of the results, we perform the calculation in the following three alternative ways:

A.5.6.1 Alternative 1

Insert expression for position from Eq(A-71) into expression for velocity in Eq(A-65):

$$v = -\frac{1}{2} \frac{r(0) \sin(\tau) C_-^{3/2}}{(\cos(\tau) + 1) \eta^2 B_-}, \quad (A-73)$$

Using the formula:

$$\tan\left(\frac{1}{2} \tau\right) = \frac{\sin(\tau)}{\cos(\tau) + 1}, \quad (A-74)$$

we can express Eq(A-73) in a somewhat more illustrative form:

$$v = -\frac{1}{2} \frac{r(0) \tan\left(\frac{1}{2} \tau\right) C_-^{3/2}}{\eta^2 B_-}, \quad (A-75)$$

A.5.6.2 Alternative 2

Integrate expression for acceleration in Eq(A-72) with respect to t (with $v=0$ for $t=0$):

$$v = \int a dt$$

$$v = \int -\frac{1}{2} \frac{r(0) C_-^3}{(\cos(\tau) + 1)^2 \eta^4 B_-^2} dt, \quad (A-76)$$

Differentiate expression for time in Eq(A-62) to change independent variable in integration from t to τ :

$$\frac{\partial}{\partial \tau} t(\tau) = \frac{\eta^2 B_- (\cos(\tau) + 1)}{C_-^{3/2}}$$

if the integral becomes:

$$v = \int -\frac{1}{2} \frac{r(0) C_-^{3/2}}{\eta^2 B_- (\cos(\tau) + 1)} d\tau$$

which integrates to:

$$v = -\frac{1}{2} \frac{r(0) \tan\left(\frac{1}{2} \tau\right) C_-^{3/2}}{\eta^2 B_-}, \quad (A-77)$$

and which is identical to the expression in Eq(A-75).

A.5.6.3 Alternative 3

Differentiate parametric expression for position with respect to t :

$$v = \frac{\partial}{\partial t} r(t)$$

$$v = \frac{\frac{\partial}{\partial \tau} r(\tau)}{\frac{\partial}{\partial \tau} t(\tau)}$$

Insert expressions for position from Eq(A-71) and time from Eq(A-62) as functions of τ :

$$v = \frac{\frac{\partial}{\partial \tau} \frac{1}{2} r(0) (\cos(\tau) + 1)}{\frac{\partial}{\partial \tau} \frac{\eta^2 B_- (\sin(\tau) + \tau)}{C_-^{3/2}}}$$

and differentiate to get:

$$v = - \frac{r(0) \sin(\tau) C_-^{3/2}}{2 \eta^2 B_- \cos(\tau) + 2 \eta^2 B_-}, \quad (\text{A-78})$$

As with Eq(A-75) above, we can express Eq(A-78) in the somewhat simpler form:

$$v = - \frac{1}{2} \frac{r(0) \tan\left(\frac{1}{2} \tau\right) C_-^{3/2}}{\eta^2 B_-}, \quad (\text{A-79})$$

which is identical to the expressions in Eq(A-77) and Eq(A-73).

A.5.8 Corollary: Constraints on τ , η , A_- , B_- , and C_- for stationary solutions

Setting:

$$\tau = x + i y, \quad (\text{A-80})$$

in Eq(A-71) we get the following expression for the position $r(\tau)$ of a volume element of the charge configuration as a function of time:

$$r(x + i y) = \frac{1}{2} r(0) (\cos(x) \cosh(y) + 1) - \frac{1}{2} i r(0) \sin(x) \sinh(y), \quad (\text{A-81})$$

In order for the position to be real, we must thus require that either $\sin(x)$ or $\sinh(y)$ be equal to zero, which corresponds to τ being purely real or purely imaginary (in which we include the special case when y is nonzero and x is a multiple of π since a constant real part of t can be included in the constant of integration $_C2$ in Eq(A-48)). From Eq(A-62) and Eq(A-71) we further conclude that in the case that τ is purely imaginary, then $r(t)$ is a monotonic function of t , and this thus corresponds to a trivial expanding solution. In the present context of stationary solutions we thus study only the nontrivial case when τ is purely real and we have a positive direction of time, *i e*

when the parameter

τ is REAL and POSITIVE.

From the relationship in Eq(A-46) between the two real quantities total charge $Q(r,t)$ and charge density $q(r,t)$ follows that the constant η has to be a (positive or negative) real number. In order for Eq(A-62) to give real values for the time t , we must then require that the quotient

$$\frac{B_-}{C_-^{3/2}}$$

is REAL, and hence its square

$$\frac{B_-^2}{C_-^3}$$

is POSITIVE.

A further constraint on η is obtained as follows. Expressing the mass density ρ in Eq(A-38) in terms of the total charge $Q(r,t)$ by means of Eq(A-45), and introducing also $T(t)$ from Eq(A-63), we obtain the following expression for the inertial mass density:

$$\rho = \frac{1}{16} \frac{\eta^5 Q(r,t)^2 B_-^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon C_-^3}, \quad (\text{A-84a})$$

From this relationship we conclude that since $Q(r,t)$ and τ are real, and since $B_-^2 / C_-^3 > 0$ in accordance with the discussion above, then in order for the inertial mass density in Eq(A-84a) to be positive we must require that the constant

η is a REAL and POSITIVE.

(This constraint on η also avoids a repetitive singularity in the time dependence of $Q(r,t)$, $q(r,t)$ and $E(r,t)$ when $\cos(\tau) = -1$ as can be seen from Eq(A-88), Eq(A-46) and Eq(A-89).)

Due to the separation of $E(r,t)$ into one radial factor $R(r)$ and one time-dependent factor $T(t)$ postulated in Eq(A-6), the scaling of $T(t)$ relative to $R(r)$ is arbitrary and we can, e g, normalize $T(t)$ to be equal to 1 when $\cos(\tau) = 1$, i e from Eq(A-63) we then have

$$B_- = \frac{1}{2} \frac{C_-}{\eta}, \quad (A-82)$$

Inserting Eq(A-82) into Eq(A-62) we get (NB sign in Eq(A-82) arbitrary, cf Eq(A-48)):

$$t = \frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}}, \quad (A-83)$$

from which we then conclude that

C_- is REAL AND POSITIVE.

Inserting Eq(A-82) into Eq(A-84a), we obtain for the inertial mass density:

$$\rho = \frac{1}{64} \frac{\eta^3 Q(r, t)^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon C_-}, \quad (A-84)$$

Inserting Eq(A-21), Eq(A-29), Eq(A-63), and Eq(A-82) into Eq(A-6), the electric field $E(r,t)$ becomes:

$$E = \frac{r^{(-\eta)} \eta \eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^2}, \quad (A-85)$$

from which we further conclude that the constant

A_- is REAL (POSITIVE or NEGATIVE).

Under the constraints set forth above, equations Eq(A-71) and Eq(A-83) thus give the following parametric expression for the position of a charge element as a function of time:

$$r(\tau) = \frac{1}{2} r(0) (\cos(\tau) + 1), \quad (\text{A-71})$$

$$t = \frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}}, \quad (\text{A-83})$$

which is the parametric representation of the common cycloid (Courant Vol I, p 261), and thus represents a stationary state of radial oscillations of the charge configuration.

A.5.9 Alternative derivation of the cycloid motion from Eq(A-47)

As a further check of the consistency in the above calculations, an alternative derivation directly from the time-dependent equation Eq(A-47) is here made of the cycloid motion described by Eq(A-71) and Eq(A-83) above. This alternative derivation also sheds light on why the motion obtained in the present electro-dynamic case is identical to the motion occurring in general relativity, e g, in the treatment of the Friedmann "dust-filled universe", as will be further discussed in Sect A.5.10 below.

From Eq(A-23) we have the following relationship for the velocity $v(r,t)$:

$$v = \frac{(r - {}_C1 \eta) \left(\frac{\partial}{\partial t} T \right)}{\eta T}, \quad (\text{A-23})$$

Since ${}_C1=0$ according to Eq(A-29) we have for a co-moving volume element, i e for $r = r(t)$:

$$\frac{\partial}{\partial t} r(t) = \frac{r(t) \left(\frac{\partial}{\partial t} T \right)}{\eta T}, \quad (\text{A-23a})$$

i e

$$\frac{\frac{\partial}{\partial t} T}{T} = \frac{\left(\frac{\partial}{\partial t} r(t)\right) \eta}{r(t)}, \quad (A-23b)$$

Integrating we obtain:

$$\ln(T) = \ln(r(t)) \eta + \ln(CI_-)$$

or

$$T = r(t)^\eta CI_-, \quad (A-23c)$$

Inserting Eq(A-23c) into the time dependent equation for $T(t)$, Eq(A-47), we get:

$$-\frac{\frac{\partial^2}{\partial t^2} r(t)^\eta CI_-}{r(t)^\eta CI_-} + \frac{\left(\frac{\partial}{\partial t} r(t)^\eta CI_-\right)^2 (\eta - 1)}{\eta \left(r(t)^\eta\right)^2 CI_-^2} = B_- \left(r(t)^\eta CI_-\right)^{\left(-3 \frac{1}{\eta}\right)}$$

i e

$$-\frac{\left(\frac{\partial^2}{\partial t^2} r(t)\right) \eta}{r(t)} = \frac{B_- CI_- \left(-3 \frac{1}{\eta}\right)}{r(t)^3}$$

or with the constant of integration CI_- redefined through $B_- = \frac{1}{2} \eta r(0) CI_- \left(3 \frac{1}{\eta}\right)$:

$$\left(\frac{\partial^2}{\partial t^2} r(t)\right) + \frac{1}{2} \frac{r(0)}{r(t)^2} = 0, \quad (A-47a)$$

Integrating Eq(A-47a) once, we obtain:

$$\left(\frac{\partial}{\partial t} r(t)\right)^2 - \frac{r(0)}{r(t)} = C2_-, \quad (A-47b)$$

as is easily checked by differentiating Eq(A-47b) with respect to t :

$$\left(\frac{\partial^2}{\partial t^2} r(t) \right) + \frac{1}{2} \frac{r(0)}{r(t)^2} = 0$$

Now, Eq(A-47b) is an equation which is well-known from gravitational theory, e.g., from the treatment of the "dust-filled Friedmann universe", viz the equation (with the constant of integration $C2_{-}=-1$) describing the self-consistent time development of a spherically-symmetric universe with uniform density and in which pressure is neglected. This Friedmann equation is known to have a solution of the form (Misner, Thorn, Wheeler: *Gravitation*, Box 27.1):

$$r(t) = \frac{1}{2} r(0) (\cos(\tau) + 1)$$

$$t = \frac{1}{2} r(0) (\sin(\tau) + \tau)$$

as is easily checked by back substitution to satisfy Eq(A-47b) (with the constant of integration $C2_{-}=-1$):

$$\frac{\sin(\tau)^2}{(\cos(\tau) + 1)^2} - 2 \frac{1}{\cos(\tau) + 1} = -1$$

i e

$$-1 = -1$$

and of course also satisfies the second-order equation, Eq(A-47a):

$$0 = 0$$

i e

$$0 = 0$$

This similarity between the electrodynamic and gravitational motion will be shown in Sect A.5.10 next to be due to fundamental similarities in the properties of the interactions in the two cases. As a prelude to that discussion, some further properties of the time-dependent equation, Eq(A-47), and its solutions will be discussed first in the next Section.

A.5.10 Another class of solutions to Eq(A-47) - gravitational analogue

Above in Sect A.5.8 we discussed the constraints which have to be imposed on the various separation constants and constants of integration in the solutions in order for the solutions to be in a physically permissible regime for the electrodynamic system under consideration. In particular, we obtained ever-expanding solutions with $\eta < 0$ ("flatter-than-Coulomb") as expected in the case when the charge distribution was unipolar and thus repulsive. On the other hand, we obtained stationary, radial oscillations with $\eta > 0$ ("steeper-than-Coulomb") in the case when the oscillations of the charge distribution produced, through polarisation, a time-varying central core of opposite charge to the surrounding unipolar charge distribution. In this case, the central core exerts an attraction on the surrounding charge distribution, which results in the radial oscillations described by the solutions with $\eta > 0$.

Mathematically, however, that is not the only case when radial oscillations appear as solutions to the time-dependent equation, Eq(A-47). To show this, consider the following set of transformations in Eq(A-47):

$$\eta = -\eta$$

$$B_- = -B_-$$

$$T = \frac{1}{T}$$

Applying these transformations to Eq(A-47), we obtain:

$$-\left(\frac{\partial^2}{\partial t^2} \frac{1}{T}\right) T - \frac{\left(\frac{\partial}{\partial t} \frac{1}{T}\right)^2 (-\eta - 1) T^2}{\eta} = -B_- \left(\frac{1}{T}\right)^{\left(3 \frac{1}{\eta}\right)}$$

or

$$-\frac{\frac{\partial^2}{\partial t^2} T}{T} + \frac{\left(\frac{\partial}{\partial t} T\right)^2 (\eta - 1)}{\eta T^2} = B_- T^{\left(-3 \frac{1}{\eta}\right)}$$

which is obviously identical to the original Eq(A-47).

Hence, in addition to the radially oscillating "steeper-than-Coulomb" solutions with $\eta > 0$ and an attracting central core ($B_- > 0$), which is the only physically permissible regime for electrodynamic confinement as derived in Sect A.5.8, we thus also have seemingly unphysical, "flatter-than-Coulomb" solutions with $\eta < 0$ and unipolar charge distribution regular at $r=0$, and which charge distribution is attractive ($B_- < 0$) even though it is unipolar.

However, these solutions are not quite as unphysical as they may seem. Going back to the assumptions we note that these solutions correspond to a force field which together with its sources obey Gauss' theorem (Eq(A-1)), and which sources obey a continuity equation (Eq(A-4)), and also momentum conservation (Eq(A-5)), but with an attractive force from a unipolar distribution.

Since these properties are characteristic of a gravitational force field, it is not unexpected that self-consistent motion in a gravitational field would turn out to closely correspond to the "unphysical", unipolar, attractive, flatter-than-Coulomb electrodynamic case discussed above, and indeed it turns out to obey exactly the same time-dependent differential equation.

In this way, it would also seem expected that cycloid trajectories would result in gravitational theory from all power-type mass distributions, not only the

> *uniform distribution assumed in the Friedmann model.*

A.6 SUMMARY

Expressed in the positive, real-valued parameter τ , the positive or negative real constant A_- and the positive real constants C_- and η according to Sect A.5.8, the previous results can be summarized as follows:

Time:

$$t = \frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}}, \quad (\text{A-83})$$

Total charge inside radius r (Eq(A-63) into Eq(A-40)):

$$Q(r, t) = 4 \frac{\pi \epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^\eta}, \quad (\text{A-86})$$

Charge of central core (Eq(A-63) into Eq(A-43)):

$$Q_o = 4 \frac{\pi \epsilon \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta A_- \eta^\eta}{\delta^\eta}, \quad (\text{A-87})$$

Charge of spherical shell outside central core (from Eq(A-44)):

$$Q_s = Q(r, t) - Q_o, \quad (\text{A-88})$$

Electric field:

$$E = \frac{r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^2}, \quad (\text{A-85})$$

$$E = \frac{1}{4} \frac{Q(r, t)}{\pi \epsilon r^2}, \quad (\text{A-89})$$

Acceleration (from Eq(A-66) alt Eq(A-72)):

$$a = -4 \frac{r C_-}{\eta^2 (\cos(\tau) + 1)^3}, \quad (A-90)$$

$$a = -2 \frac{r(0) C_-}{(\cos(\tau) + 1)^2 \eta^2}, \quad (A-91)$$

Velocity (from Eq(A-65) alt Eq(A-75)):

$$v = -2 \frac{r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2}, \quad (A-92)$$

$$v = - \frac{r(0) \tan\left(\frac{1}{2} \tau\right) \sqrt{C_-}}{\eta}, \quad (A-93)$$

Position:

$$r(\tau) = \frac{1}{2} r(0) (\cos(\tau) + 1), \quad (A-71)$$

Charge density (Eq(A-86) into Eq(A-46)):

$$q = - \frac{\epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2}\right)^\eta \eta}{r \eta r^3}, \quad (A-94)$$

$$q = - \frac{1}{4} \frac{Q(r, t) \eta}{\pi r^3}, \quad (A-46)$$

Mass density (Eq(A-86) into Eq(A-84)):

$$\rho = 2 \frac{\eta^{(3+2\eta)} \epsilon A_-^2 \left(\frac{1}{2} \cos(\tau) + \frac{1}{2}\right)^{(3+2\eta)}}{r^{(6+2\eta)} C_-}, \quad (A-95)$$

$$\rho = \frac{1}{64} \frac{\eta^3 Q(r, t)^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon C_-}, \quad (A-84)$$

A.6.1 Simplification by Redefinition of Constants (cf preprint aps1997may13_001)

The constants A_- , C_- , and η introduced as constants of integration and separation in the above solutions of the electrodynamic equations can be redefined to give somewhat simpler expressions of the solutions.

Defining an angular frequency ω such that

$$\omega t = 2 \pi$$

for

$$\tau = 2 \pi$$

we get from Eq(A-83):

$$2 \frac{\pi}{\omega} = \frac{\eta \pi}{\sqrt{C_-}}$$

i e

$$C_- = \frac{1}{4} \eta^2 \omega^2$$

It is also practical to redefine the scaling constant A_- as:

$$A_- = \frac{1}{4} \frac{A^2 \eta}{\pi \epsilon \eta}$$

Thus, introducing the

$$\text{substitutions} := C_- = \frac{1}{4} \eta^2 \omega^2, A_- = \frac{1}{4} \frac{A^2 \eta}{\pi \epsilon \eta}$$

into the solutions above we obtain the following simplified expressions (the equation numbers are taken from preprint aps1997may13_001):

From Eq(A-89):

$$E = \frac{1}{4} \frac{Q(r, t)}{\pi \epsilon r^2}, \quad \text{Eq(7)}$$

From Eq(A-92):

$$v = -2 \frac{r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2}$$
$$v = -\frac{1}{2} \frac{r \sin(\tau) \sqrt{4} \omega}{(\cos(\tau) + 1)^2}$$

i e

$$v = -\frac{r \sin(\tau) \omega}{(\cos(\tau) + 1)^2}, \quad \text{Eq(8)}$$

From Eq(A-46):

$$q = -\frac{1}{4} \frac{Q(r, t) \eta}{\pi r^3}, \quad \text{Eq(9)}$$

From Eq(A-84):

$$\rho = \frac{1}{64} \frac{\eta^3 Q(r, t)^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon C_-}$$
$$\rho = \frac{1}{16} \frac{\eta Q(r, t)^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon \omega^2}$$

i e

$$\rho = \frac{1}{16} \frac{\eta Q(r, t)^2 (\cos(\tau) + 1)^3}{r^6 \pi^2 \epsilon \omega^2}, \quad \text{Eq(10)}$$

From Eq(A-83):

$$t = \frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}}$$

$$t = \frac{1}{2} \frac{\sqrt{4} (\sin(\tau) + \tau)}{\omega}$$

i e

$$t = \frac{\sin(\tau) + \tau}{\omega}, \quad \text{Eq(11)}$$

From Eq(A-86):

$$Q(r, t) = 4 \frac{\pi \varepsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^\eta}$$

$$Q(r, t) = A r^{(-\eta)} (\cos(\tau) + 1)^\eta, \quad \text{Eq(12)}$$

From Eq(A-87):

$$Q_o = 4 \frac{\pi \varepsilon \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta A_- \eta^\eta}{\delta^\eta}$$

$$Q_o = \delta^{(-\eta)} A (\cos(\tau) + 1)^\eta, \quad \text{Eq(13)}$$

From Eq(A-71):

$$r(\tau) = \frac{1}{2} r(0) (\cos(\tau) + 1), \quad \text{Eq(14)}$$

From Eq(18):

$$\omega_{[p]}^2 = \frac{q^2}{\varepsilon \rho}$$

$$\omega_{[p]}^2 = \frac{\eta \omega^2}{(\cos(\tau) + 1)^3}$$

i e

$$\omega_{[p]} = \frac{\sqrt{\eta} \omega}{(\cos(\tau) + 1)^{3/2}},$$

Eq(19)

>

A.7 VERIFICATION

The consistency of the results has been concurrently verified in the derivations above. As a final check of the consistency of the results, the expressions for the quantities listed in Sect A.6 SUMMARY are here substituted back into the equations listed in Sect A.1 POSTULATES:

A.7.1 Back Substitution into Maxwell's Equation for Charge Density, Eq(A-1)

Insert expressions for charge density $q(r,t)$ from Eq(A-94) and field $E(r,t)$ from Eq(A-85) into Maxwell's equation Eq(A-1):

$$-\frac{\epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \eta}{r^\eta r^3} = \epsilon \left(2 \frac{r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^3} + \left(\frac{\partial}{\partial r} \frac{r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^2} \right) \right)$$

and simplify:

$$-4 \frac{\pi \epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \eta}{r r^\eta} = -4 \frac{\pi \epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \eta}{r r^\eta}$$

The expressions in SUMMARY are thus consistent with the Maxwell equation Eq(A-1).

A.7.2 Back Substitution into Maxwell's Equation for Current Density, Eq(A-2)

Insert expressions for charge density $q(r,t)$ from Eq(A-94) and velocity $v(r,t)$ from

Eq(A-92) into Maxwell's equation Eq(A-2):

$$2 \frac{\varepsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \sin(\tau) \sqrt{C_-}}{r^\eta r^2 (\cos(\tau) + 1)^2} + \varepsilon \left(\frac{\partial}{\partial t} E \right) = 0, \quad (A-96)$$

Change from differentiation of $E(r,t)$ with respect to time t to differentiation with respect to τ using Eq(A-85) and Eq(A-83):

$$\frac{\partial}{\partial t} E = \frac{\frac{\partial}{\partial \tau} \frac{r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^2}}{\frac{\partial}{\partial \tau} \frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}}}$$

i e

$$\frac{\partial}{\partial t} E = - \frac{r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \sin(\tau) \sqrt{C_-}}{r^2 \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right) (\cos(\tau) + 1)}$$

and Eq(A-96) can thus be rewritten as:

$$2 \frac{\varepsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \sin(\tau) \sqrt{C_-}}{r^\eta r^2 (\cos(\tau) + 1)^2} - 2 \frac{\varepsilon r^{(-\eta)} \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \sin(\tau) \sqrt{C_-}}{r^2 (\cos(\tau) + 1)^2} = 0$$

i e

$$0 = 0$$

The expressions in SUMMARY are thus consistent with the Maxwell equation Eq(A-2).

A.7.3 Back Substitution into Continuity Equation for Charge Density, Eq(A-3)

Change from differentiation of q with respect to time t to differentiation with respect to τ in the continuity equation for charge density Eq(A-3) using Eq(A-94) and Eq(A-83):

$$\frac{\epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \eta \sin(\tau) \sqrt{C_-}}{r^\eta \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right) r^3 (\cos(\tau) + 1)} + 2 \frac{q v}{r} + \left(\frac{\partial}{\partial r} q \right) v + q \left(\frac{\partial}{\partial r} v \right) = 0, \quad (A-97)$$

Insert the expression for $q(r, \tau)$ from Eq(A-94) and the expression for $v(r, \tau)$ from Eq(A-92):

$$\frac{\epsilon \eta^\eta A_- \%1^\eta \eta \sin(\tau) \sqrt{C_-}}{r^\eta \%1 r^3 (\cos(\tau) + 1)} + 4 \frac{\epsilon \eta^\eta A_- \%1^\eta \sin(\tau) \sqrt{C_-}}{r^3 r^\eta (\cos(\tau) + 1)^2} - 2 \frac{\left(\frac{\partial}{\partial r} - \frac{\epsilon \eta^\eta A_- \%1^\eta \eta}{r^\eta r^3} \right) r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} - \frac{\epsilon \eta^\eta A_- \%1^\eta \eta \left(\frac{\partial}{\partial r} - 2 \frac{r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} \right)}{r^\eta r^3} = 0$$

$$\%1 := \frac{1}{2} \cos(\tau) + \frac{1}{2}$$

and simplify:

$$\frac{\epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta \eta \sin(\tau) \sqrt{C_-}}{r^\eta \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right) r^3 (\cos(\tau) + 1)} - 2 \frac{\eta \sin(\tau) \sqrt{C_-} \epsilon \eta^\eta A_- \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^\eta}{r^3 (\cos(\tau) + 1)^2 r^\eta} = 0$$

ie

$$0 = 0$$

The expressions in SUMMARY are thus consistent with the continuity equation for charge density, Eq(A-3). A further verification of the consistency of the above solution is given in Sect B.4 where it is shown that the charge within a co-moving radius is constant in time.

A.7.4 Back Substitution into Continuity Equation for Mass Density, Eq(A-4)

Change from differentiation of ρ with respect to time t to differentiation with respect to τ in the continuity equation for mass density Eq(A-4) using Eq(A-95) and Eq(A-83):

$$\frac{\frac{\partial}{\partial \tau} \left[\frac{\eta^{(3+2\eta)} \epsilon A_-^2 \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^{(3+2\eta)}}{r^{(6+2\eta)} C_-} \right]}{\frac{\partial}{\partial \tau} \left[\frac{1}{2} \frac{\eta (\sin(\tau) + \tau)}{\sqrt{C_-}} \right]} + \frac{\frac{\partial}{\partial r} r^2 \rho v}{r^2} = 0$$

ie

$$-2 \frac{\eta^{(3+2\eta)} \epsilon A_-^2 \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^{(3+2\eta)} (3+2\eta) \sin(\tau)}{\left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right) r^{(6+2\eta)} \sqrt{C_-} \eta (\cos(\tau) + 1)} + 2 \frac{\rho v}{r} + \left(\frac{\partial}{\partial r} \rho \right) v + \rho \left(\frac{\partial}{\partial r} v \right) = 0, \quad (A-98)$$

Insert the expression for $\rho(r, \tau)$ from Eq(A-95) and the expression for $v(r, \tau)$ from Eq(A-92) into Eq(A-98):

$$-2 \frac{\epsilon A_-^2 (3+2\eta) \sin(\tau)}{\sqrt{C_-} \eta (\cos(\tau) + 1)} - 8 \frac{\epsilon A_-^2 \sin(\tau)}{\sqrt{C_-} \eta (\cos(\tau) + 1)^2} - 2 \frac{\left(\frac{\partial}{\partial r} \frac{\epsilon A_-^2 (3+2\eta)}{C_-} \right) r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} + 2 \frac{\left(\frac{\partial}{\partial r} - 2 \frac{r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} \right)}{C_-} = 0$$

$$\%1 := r^{(6+2\eta)}$$

$$\%2 := \frac{1}{2} \cos(\tau) + \frac{1}{2}$$

$$\%3 := \%2^{(3+2\eta)}$$

$$\%4 := \eta^{(3+2\eta)}$$

Evaluate :

$$\begin{aligned} & -2 \frac{\eta^{(3+2\eta)} \epsilon A_-^2 \%1^{(3+2\eta)} (3+2\eta) \sin(\tau)}{\%1 r^{(6+2\eta)} \sqrt{C_-} \eta (\cos(\tau) + 1)} \\ & - 12 \frac{\eta^{(3+2\eta)} \epsilon A_-^2 \%1^{(3+2\eta)} \sin(\tau)}{r^{(6+2\eta)} \sqrt{C_-} \eta (\cos(\tau) + 1)^2} \\ & + 4 \frac{\eta^{(3+2\eta)} \epsilon A_-^2 \%1^{(3+2\eta)} (6+2\eta) \sin(\tau)}{r^{(6+2\eta)} \sqrt{C_-} \eta (\cos(\tau) + 1)^2} = 0 \end{aligned}$$

$$\%1 := \frac{1}{2} \cos(\tau) + \frac{1}{2}$$

and simplify:

$$0 = 0$$

The expressions in SUMMARY are thus consistent with the continuity equation for mass density, Eq(A-4). As a further verification of the consistency of the above solution it will be shown in Sect B.5 that the mass within a co-moving spherical shell is constant in time.

A.7.5 Back Substitution into Newton's Second Law, Eq(A-5)

Change from differentiation of v in Eq(A-92) with respect to time t to differentiation with respect to τ in Newton's second law Eq(A-5) using Eq(A-83):

$$2 \frac{\left(-2 \frac{r \cos(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} - 4 \frac{r \sin(\tau)^2 \sqrt{C_-}}{\eta (\cos(\tau) + 1)^3} \right) \sqrt{C_-}}{\eta (\cos(\tau) + 1)} + v \left(\frac{\partial}{\partial r} v \right) = \frac{q E}{\rho}, \quad (A-99)$$

Insert expressions for velocity $v(r,t)$ from Eq(A-92), charge density $q(r,t)$ from Eq(A-94), mass density $\rho(r, t)$ from Eq(A-95), and electric field $E(r,t)$ from Eq(A-85):

$$\begin{aligned}
& 2 \frac{\left(-2 \frac{r \cos(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} - 4 \frac{r \sin(\tau)^2 \sqrt{C_-}}{\eta (\cos(\tau) + 1)^3} \right) \sqrt{C_-}}{\eta (\cos(\tau) + 1)} \\
& - 2 \frac{r \sin(\tau) \sqrt{C_-} \left(\frac{\partial}{\partial r} - 2 \frac{r \sin(\tau) \sqrt{C_-}}{\eta (\cos(\tau) + 1)^2} \right)}{\eta (\cos(\tau) + 1)^2} = \\
& - \frac{1}{2} \frac{(\eta \eta)^2 \left(\left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right) \eta \right)^2 \eta r^{(-\eta)} r^{(6+2\eta)} C_-}{r \eta r^5 \eta^{(3+2\eta)} \left(\frac{1}{2} \cos(\tau) + \frac{1}{2} \right)^{(3+2\eta)}}
\end{aligned}$$

which simplifies to:

$$-4 \frac{C_- r}{\left(\cos(\tau)^3 + 3 \cos(\tau)^2 + 3 \cos(\tau) + 1 \right) \eta^2} = -4 \frac{C_- r}{\left(\cos(\tau)^3 + 3 \cos(\tau)^2 + 3 \cos(\tau) + 1 \right) \eta^2}$$

The expressions in SUMMARY are thus consistent with Newton's second law Eq(A-5).

>

A.8 CONSISTENCY WITH THE VIRIAL THEOREM

If correct, the radial oscillations derived in this Appendix have to be consistent also with the virial theorem, which is a general relationship which dynamic systems must obey. In this Section, the results summarized in Sect A.6 will be inserted into the virial theorem in order to show that these solutions are consistent with a stationary system (which is to be expected, since the virial theorem is based on the fact that the divergence of the total stress-energy tensor for matter plus electromagnetic field is zero, which is equivalent to the basic assumptions Eq(A-1) through Eq(A-5) made in the derivation of the present solutions).

A.8.1 The virial theorem

*The virial theorem due to Chandrasekhar & Fermi relates $d^2 J(t) / dt^2$, the second time-derivative of the radial moment of inertia, for a system to a sum of kinetic, potential and electromagnetic quantities for the system. The relativistic derivation of Rosenbluth & Stuart in *Phys Fluids* 6, p 452 (1963) is usually taken as the starting point for the analysis of electromagnetic systems in the form of the following expressions (in Gaussian units):*

With the radial moment of inertia $J(t)$ defined as:

$$J(t) = \int r^2 \left(\frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{1}{8} \frac{E^2 + H^2}{\pi c^2} \right) dV, \quad (A-100a)$$

the (time-dependent) virial theorem is given as:

$$\begin{aligned}
\frac{d^2 J(t)}{dt^2} = & 2 \int \frac{\rho v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{1}{8} \frac{E^2 + H^2}{\pi} dV - \int \frac{1}{4} \frac{(E^2 + H^2) r \cdot}{\pi} dS \\
& + \int \frac{1}{2} \frac{(E(r.E) + H(r.H)) \cdot}{\pi} dS - \frac{d}{dt} \int r^2 \left(\frac{1}{4} \frac{E \times H}{\pi c} + \frac{\rho v}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \cdot dS, \tag{A-100b}
\end{aligned}$$

Systems for which $d^2 J(t) / dt^2$ is positive-definite or negative-definite correspond to systems which are ever expanding or collapsing, respectively. By averaging $d^2 J(t) / dt^2$ over long time, the ordinary (time-independent) virial theorem is obtained which states that $\langle d^2 J(t) / dt^2 \rangle = 0$ for stable systems.

In systems dominated by electromagnetic forces, the total energy corresponding to the volume integral on the right hand side of Eq(A-100) is positive. Also, the surface integrals are purported to vanish through evaluating them at infinity. Hence, the second time-derivative of the radial moment of inertia according to Eq(A-100) - here also augmented with the inertia due to the energy of the electromagnetic field - is alleged to be positive-definite. This result is then used to conclude that no self-confined system held together only by electromagnetic forces can ever be stable (Schmidt, *Phys Fluids* 3, p 481 (1960), cf also the above work by Rosenbluth & Stuart).

The present Section is aimed at showing that this conclusion is not general, and that the radial oscillations derived in this Appendix for "steeper-than-Coulomb" systems ($\eta > 0$) do constitute a counterexample to Schmidt's conclusion. Why this is so is essentially due to the step above when the surface integrals in Eq(A-100) are set to zero. This is not true for the present steep functions with $\eta > 0$, where instead the electromagnetic pressure and impulse contributions described by these surface integrals at the inner boundary may dominate the virial balance, as will be shown

in the following.

In a nonrelativistic, purely electrodynamic system, i e with

$$H = 0, \frac{v^2}{c^2} = 0$$

the radial moment of inertia becomes:

$$J(t) = \int r^2 \left(\rho + \frac{1}{8} \frac{E^2}{\pi c^2} \right) dV, \quad (A-101a)$$

and the virial theorem becomes:

$$\frac{d^2 J(t)}{dt^2} = 2 \int \rho v^2 + \frac{1}{8} \frac{E^2}{\pi} dV - \int \frac{1}{4} \frac{E^2 r \cdot}{\pi} dS + \int \frac{1}{2} \frac{E (r \cdot E) \cdot}{\pi} dS - \frac{d}{dt} \int r^2 \rho v \cdot dS, \quad (A-101b)$$

In MKSA units as used above in this Appendix, the radial moment of inertia and the virial theorem become:

$$J(t) = \int r^2 \left(\rho + \frac{1}{2} \frac{E^2 \epsilon}{c^2} \right) dV, \quad (A-102a)$$

$$\frac{d^2 J(t)}{dt^2} = 2 \int \rho v^2 + \frac{1}{2} E^2 \epsilon dV - \int E^2 \epsilon r \cdot dS + \int 2 E \epsilon (r \cdot E) \cdot dS - \frac{d}{dt} \int r^2 \rho v \cdot dS, \quad (A-102b)$$

In spherical symmetry the E field is always directed in the r direction, i e

$$E(r.E) = E^2 r, \quad (A-103)$$

and the virial theorem in Eq(A-103) then becomes:

$$\frac{d^2 J(t)}{dt^2} = \int 2 \rho v^2 + E^2 \epsilon dV + \int E^2 \epsilon r \cdot dS - \frac{d \int r^2 \rho v \cdot dS}{dt}, \quad (A-104)$$

A.8.2 Insertion of solutions from Sect A.6

For clarity, the following calculations will be made for one term of the right hand side of Eq(A-104) at a time, viz

$$\int 2 \rho v^2 dV = \left[\int 8 \rho v^2 \pi r^2 dr \right]^{R_-} - \left[\int 8 \rho v^2 \pi r^2 dr \right]^{\delta}, \quad (A-105a)$$

$$\int E^2 \epsilon dV = \left[\int 4 E^2 \epsilon \pi r^2 dr \right]^{R_-} - \left[\int 4 E^2 \epsilon \pi r^2 dr \right]^{\delta}, \quad (A-105b)$$

$$\int E^2 \epsilon r \cdot dS = \left[4 E^2 \epsilon r^3 \pi \right]^{R_-} - \left[4 E^2 \epsilon r^3 \pi \right]^{\delta}, \quad (A-105c)$$

$$-\frac{d \int r^2 \rho v \cdot dS}{dt} = -\frac{d \left[4 r^4 \rho v \pi \right]^{R_-}}{dt} + \frac{d \left[4 r^4 \rho v \pi \right]^{\delta}}{dt}, \quad (A-105d)$$

where the square brackets indicate that the expression inside is to be evaluated at the inner boundary δ or outer boundary R_- as indicated at the upper right corner.

From the expressions in Eq(7), Eq(8), Eq(10) and Eq(12) we conclude that the electric field E , the velocity v and inertial mass density ρ depend on r as follows:

$$\begin{aligned}
E &\sim r^{(-\eta-2)} \\
v &\sim r \\
\rho &\sim r^{(-6-2\eta)}
\end{aligned}$$

so that for a system extending from $\delta \rightarrow 0$ to $R_- \rightarrow \infty$ some terms in Eq(A-105a) through Eq(A-105d) vanish, leaving the following expressions:

$$\int 2 \rho v^2 dV = - \left[\int 8 \rho v^2 \pi r^2 dr \right]^\delta, \quad (A-106a)$$

$$\int E^2 \varepsilon dV = - \left[\int 4 E^2 \varepsilon \pi r^2 dr \right]^\delta, \quad (A-106b)$$

$$\int E^2 \varepsilon r \cdot dS = - \left[4 E^2 \varepsilon r^3 \pi \right]^\delta, \quad (A-106c)$$

$$- \frac{d \int r^2 \rho v \cdot dS}{dt} = \frac{d \left[4 r^4 \rho v \pi \right]^\delta}{dt}, \quad (A-106d)$$

After inserting Eq(7), Eq(8) and/or Eq(10) from Section A.6 into Eq(A-106a) through Eq(A-106d) and using Eq(12), we get for the different terms in the virial theorem, Eq(A-104), after evaluation:

$$\begin{aligned}
\int 2 \rho v^2 dV &= - \left[\int \frac{\frac{1}{2} \eta A^2 \left(r^{(-\eta)} \right)^2 \left((\cos(\tau) + 1) \eta \right)^2 \sin(\tau)^2}{r^2 \pi \varepsilon (\cos(\tau) + 1)} dr \right]^\delta \\
&= - \frac{\left(-\frac{1}{2} + \frac{1}{2} \cos(\tau) \right) (\cos(\tau) + 1)^{(2\eta)} A^2 \eta \delta^{(-2\eta-1)}}{(2\eta+1) \pi \varepsilon}, \\
\end{aligned} \quad (A-107a)$$

$$\int E^2 \varepsilon dV = \left[\int_0^\delta \frac{1}{4} \frac{A^2 (r^{(-\eta)})^2 ((\cos(\tau) + 1)\eta)^2}{\pi \varepsilon r^2} dr \right]$$

$$= \frac{1}{4} \frac{A^2 (\cos(\tau) + 1)^{(2\eta)} \delta^{(-2\eta - 1)}}{\pi \varepsilon (2\eta + 1)}, \quad (A-107b)$$

$$\int E^2 \varepsilon r \cdot dS = \left[\int_0^\delta \frac{1}{4} \frac{A^2 (r^{(-\eta)})^2 ((\cos(\tau) + 1)\eta)^2}{\pi \varepsilon r} \right]$$

$$= \frac{1}{4} \frac{(\cos(\tau) + 1)^{(2\eta)} A^2 \delta^{(-2\eta - 1)}}{\pi \varepsilon}, \quad (A-107c)$$

$$\frac{d}{dt} \int r^2 \rho v \cdot dS = \frac{d}{dt} \left[\int_0^\delta \frac{1}{4} \frac{\eta A^2 (r^{(-\eta)})^2 ((\cos(\tau) + 1)\eta)^2 (\cos(\tau) + 1) \sin(\tau)}{r \pi \varepsilon \omega} \right]$$

$$= \frac{d}{dt} \left[\frac{1}{4} \frac{(\cos(\tau) + 1)^{(2\eta + 1)} A^2 \sin(\tau) \eta \delta^{(-2\eta - 1)}}{\omega \pi \varepsilon} \right], \quad (A-107d)$$

According to Eq(14), the inner boundary δ is time-dependent:

$$\delta = \frac{1}{2} \delta(0) (\cos(\tau) + 1), \quad (A-108)$$

It is also practical to redefine constants by introducing a new constant K , defined through:

$$A^2 = K 2^{(-2\eta)} \delta(0)^{(2\eta + 1)} \pi \varepsilon, \quad (A-109)$$

Introducing Eq(A-108) and Eq(A-109) into Eq(A-107a) through Eq(A-107d) we get

for the terms in the virial theorem:

$$\int 2 \rho v^2 dV = -2 \frac{\left(-\frac{1}{2} + \frac{1}{2} \cos(\tau)\right) K \eta}{(\cos(\tau) + 1)(2\eta + 1)}, \quad (A-110a)$$

$$\int E^2 \varepsilon dV = \frac{1}{2} \frac{K}{(\cos(\tau) + 1)(2\eta + 1)}, \quad (A-110b)$$

$$\int E^2 \varepsilon r \cdot dS = -\frac{1}{2} \frac{K}{\cos(\tau) + 1}, \quad (A-110c)$$

$$-\frac{d \int r^2 \rho v \cdot dS}{dt} = \frac{\partial}{\partial t} - \frac{1}{2} \frac{K \sin(\tau) \eta}{\omega}, \quad (A-110d)$$

In order to calculate the derivative with respect to time t in Eq(A-110d) above, in which time appears implicitly through the parameter τ , the partial derivative of t with respect to τ is needed, which we get from Eq(11):

$$\frac{\partial}{\partial \tau} t = \frac{\cos(\tau) + 1}{\omega}, \quad (A-111)$$

Thus we can evaluate the last term, Eq(A-110d), on the right hand side in the virial theorem as follows:

$$-\frac{d \int r^2 \rho v \cdot dS}{dt} = \frac{\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{K \sin(\tau) \eta}{\omega}}{\frac{\partial}{\partial \tau} t}$$

$$-\frac{d \int r^2 \rho v \cdot dS}{dt} = -\frac{1}{2} \frac{K \cos(\tau) \eta}{\cos(\tau) + 1}, \quad (A-110d')$$

From Eq(A-110a), Eq(A-110b), Eq(A-110c) and Eq(A-110d ') we can thus finally add up the right hand side of the virial theorem, Eq(A-104), and get:

$$\int 2 \rho v^2 dV + \int E^2 \varepsilon dV + \int E^2 \varepsilon r \cdot dS - \frac{d}{dt} \int r^2 \rho v \cdot dS =$$

$$-2 \frac{\left(-\frac{1}{2} + \frac{1}{2} \cos(\tau) \right) K \eta}{(\cos(\tau) + 1) (2 \eta + 1)} + \frac{1}{2} \frac{K}{(\cos(\tau) + 1) (2 \eta + 1)} - \frac{1}{2} \frac{K}{\cos(\tau) + 1} - \frac{1}{2} \frac{K \cos(\tau) \eta}{\cos(\tau) + 1}$$

or simplified:

$$\int 2 \rho v^2 dV + \int E^2 \varepsilon dV + \int E^2 \varepsilon r \cdot dS - \frac{d}{dt} \int r^2 \rho v \cdot dS = -\frac{1}{2} \frac{K \eta \cos(\tau) (3 + 2 \eta)}{(\cos(\tau) + 1) (2 \eta + 1)},$$

(A-104b)

For the oscillating solutions summarized in Section A.6, the virial theorem, Eq(A-104), thus gives:

$$\frac{d^2 J(t)}{dt^2} = -\frac{1}{2} \frac{K \eta \cos(\tau) (3 + 2 \eta)}{(\cos(\tau) + 1) (2 \eta + 1)}, \quad (A-112)$$

A.8.3 Verification of stationary solutions

Averaging Eq(A-112) over a period Δt we have:

$$\langle d^2 J(t) / dt^2 \rangle = \frac{\int_{t_0}^{t_0 + \Delta t} -\frac{1}{2} \frac{K \eta \cos(\tau) (3 + 2 \eta)}{(\cos(\tau) + 1) (2 \eta + 1)} dt}{\Delta t}$$

$$\langle d^2 J(t)/dt^2 \rangle = \frac{1}{2} \frac{\int_0^{2\pi} -\frac{1}{2} \frac{K \eta \cos(\tau) (3 + 2 \eta) \left(\frac{\partial}{\partial \tau} t\right)}{(\cos(\tau) + 1) (2 \eta + 1)} d\tau}{\pi}, \quad (A-113)$$

Using Eq(A-111) we get:

$$\langle d^2 J(t)/dt^2 \rangle = \frac{1}{2} \frac{\int_0^{2\pi} -\frac{1}{2} \frac{K \eta \cos(\tau) (3 + 2 \eta)}{(2 \eta + 1) \omega} d\tau}{\pi}, \quad (A-114)$$

ie

$$\langle d^2 J(t)/dt^2 \rangle = 0, \quad (A-115)$$

Inserting the solutions in Section A.6 SUMMARY into the virial theorem thus gives a result consistent with a stationary system.

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