

PHYSICAL ADDITION

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Modern physics is unthinkable without mathematics. Its conceptual apparatus is permeated by numbers. In ancient and medieval times it was a problem whether mathematics could be useful in fundamental physics. Today we face the opposite problem. Are there any interesting non-mathematical concepts in physics?

In the center of mathematics we find the natural numbers and the operation of adding these numbers. Usually, when philosophers of science have been discussing addition in physics, they have looked upon pure mathematical addition as something well-known and then explained physical addition as an *application* of mathematical addition. Thus, the direction has been from operations between numbers to operations between things. Physical addition has, in such approaches, no identity of its own. The distinction between additive and non-additive quantities gets its whole identity from mathematics. I shall try to show that this traditional view is false. There are properties which are naturally called additive (and subtractive) independently of whether these properties are quantifiable or not. When the properties in question lend themselves to quantification we get what is sometimes called extensive, sometimes additive quantities (or magnitudes).

'Top-down'and 'bottom-up'

Once upon a time in analytic philosophy, nominalist and operationist accounts of quantities reigned supreme. According to nominalism, even ordinary properties are structures which we impose, by means of our language, on particulars. Nominalism with regard to quantities, then, becomes a mere corollary. However, it is possible to be a realist with regard to properties but a nominalist with regard to quantities. A realist view of quantities says that quantities are ordinary properties that admit of degrees and that quantities, therefore, can inhere in particulars; nominalism about quantities says that even if properties can inhere in particulars, quantities have to be conceptual constructs.

In a similar way it is possible to distinguish between an all-embracing operationism and operationism with regard to quantities. General operationism claims that *all* properties are logically secondary to operations; 'quantity operationism' claims only that, even if some

properties can precede measuring operations, quantities cannot. There is a close relationship between nominalism and operationism in general as well as between quantity nominalism and quantity operationism.

Let us now, as an introduction to the distinction between additive and non-additive quantities, take a quick look at Carl Hempel's analytic philosophical classic *Fundamentals of Concept Formation in Empirical Science*. According to Hempel, quantities are functions. A function which assigns to every element, x , in a domain exactly one real number, $s(x)$, constitutes a quantity.¹ Quantities, though, as Hempel makes clear, can be of different kinds:

The distinction of additive and non-additive quantities refers to the existence or nonexistence, for a given quantitative concept, of an *operational interpretation for the numerical addition* of the s -values of two different objects. In this sense, length is called an additive quantity *because* the sum of two numerical length-values can be *represented* as the length of the interval obtained by joining two intervals of the given lengths end to end in a straight line; temperature is said to be non-additive because there is no operation on two bodies of given temperatures which will produce an object whose temperature equals the sum of the latter. To state this idea more precisely, we first define a relative concept of additivity:

(14.1) A quantity s is additive relatively to a combining operation o if $s(xoy) = s(x) + s(y)$ whenever x , y , and xoy belong to the domain within which s is defined.²

It is not quite clear whether Hempel has a realist view of the property of length or not, but even if he has (though I think he leans heavily towards a general nominalism), there is for him, obviously, nothing in the property length itself which makes it additive. It is by accident, so to speak, addition can be represented by an operation of concatenation.

Another classic in the field was Brian Ellis' *Basic Concepts of Measurement*³. Ellis criticized not only operationism about quantities but also a realist account of quantities. He regarded quantities as objective linear orders not dependent upon quantitative universals. He himself, however, has subsequently changed his mind. Now he thinks that quantities can have

¹ *Fundamentals of Concept Formation in Empirical Science*, University of Chicago Press: Chicago & London 1952, p 63.

² Op. cit. p 75, emphases added.

³ Cambridge University Press: London 1966.

magnitudes independently of our measuring operations.⁴ There are though, still, philosophers (John Forge⁵ in particular) who explicitly defend positions closely related to those of (early) Ellis.

The similarity between Hempel, (early) Ellis, and Forge is, firstly, that they defend a nominalism about quantities, and, secondly, that their position with regard to non-quantitative properties is ontologically unclear.

Mario Bunge has always explicitly been a realist with regard to properties, but, like Hempel, he regards quantities or magnitudes as functions. Bunge says that "Functions are the structure of quantitative concepts or *magnitudes*, also called quantities"⁶ and that:

In general, any magnitude M involving an object variable and a numerical variable - such as length and population density - can be analyzed as a function M from a physical set $P \times S$ to a numerical set R .⁷ (P is a set of physical objects and S a set of scales.)

Certainly, Bunge distinguishes between mathematical addition and physical addition as an operation of concatenation, but, none the less, his distinction between additive and non-additive magnitudes is founded not in the properties themselves but in a contingent similarity between concatenation and mathematical addition. He says that:

If two or more objects are considered for measurement, more than one object variable may enter the magnitude. For example, if two rods, x and y , are juxtaposed end to end, a third object z is produced which may be said to be the *physical sum* (or joining) of x and y . We have denoted physical addition by ' $+^*$ ' to distinguish it from the corresponding arithmetical operation: the former regards bodies, the latter numbers. If we now ask what the length $L(x+^*y)$ of the composite rod $z =^* x+^*y$ is, the answer will be: the numerical value of the total length equals the sum of the partial lengths, i.e.

$$L(x+^*y) = L(x) + L(y)$$

regardless of the length scale and unit.

The foregoing is a synthetic (nonlogical) formula: universes are conceivable in

⁴ See Ellis, 'Comments on Forge and Swoyer' in J. Forge (ed.) *Measurement, Realism and Objectivity*, Reidel: Dordrecht 1987, pp 319-25.

⁵ In particular, see J. Forge, 'On Ellis' Theory of Quantities' in J. Forge (ed.) *Measurement, Realism and Objectivity*, Reidel: Dordrecht 1987, pp 291-310.

⁶ Bunge, *Scientific Research*, Springer-Verlag: Berlin, Heidelberg & New York 1967, vol. I, p 61.

⁷ Op. cit. vol. I, p 70.

which lengths do not add in this simple way - i.e. in which length is not an additive measure.⁸

I think Bunge in his nominalist view of quantities moves too fast between properties realistically conceived, numbers and quantities or magnitudes. Something more has to be said about relations like 'being larger than', 'being more massive', 'being more dense', etc. Some Australian philosophers, however, have (from my point of view) tried to remedy this defect and made relations important in the analysis of quantities. David M. Armstrong, in a comment on these attempts, distinguishes between two realist approaches, a *top-down* strategy and a *bottom-up* strategy.⁹ The top-down strategy is put forward in a paper by John Bigelow and Robert Pargetter.¹⁰ They claim that quantities are in essence relations (conceived as realist universals). Chris Swoyer¹¹, and Armstrong himself, on the other hand, claims that quantities are properties. In the top-down approach relations are logically prior to quantitative properties; in the bottom-up approach properties are logically prior to relations.

Armstrong's metaphors 'top-down' and 'bottom-up' are telling. It seems natural to regard relations as in some way hovering above properties. The same is true for concepts and functions, which means that not only Bigelow and Pargetter's analysis, but also Hempel's, Ellis', Forge's, and Bunge's analyses of quantities may be called top-down procedures. All these attempts have something in common which makes them differ from bottom-up procedures. I think Armstrong has hit upon the really interesting divide in the philosophy of quantities.

I am myself confident that the bottom-up procedure is the ontologically correct one, and I shall now try to show that not only quantities, but also the distinction between additive and non-additive quantities, can be explained in a bottom-up fashion. Not only some properties and quantities, but also the distinction between additive and non-additive quantities does in some cases precede actual measurement. Some properties are in virtue of their 'nature' additive and some are not. Neither Armstrong nor Bigelow and Pargetter have really focused attention on this issue.

Properties and their relation to space

My fundamental thesis is that different properties can be *differently related to space* and that

⁸ Op. cit. vol. II, p 199.

⁹ Armstrong, 'Are quantities relations? A reply to Bigelow and Pargetter' in *Philosophical Studies* 54 (1988), pp 305-16.

¹⁰ Bigelow & Pargetter, 'Quantities' in *Philosophical Studies* 54 (1988), pp 287-304.

¹¹ Swoyer, 'The Metaphysics of Measurement' in J. Forge (ed.) *Measurement, Realism and Objectivity*, Reidel: Dordrecht 1987, pp 235-90.

their relation to space explains why some of them are naturally additive and others non-additive. Kant pointed out that the difference between a right hand and a left hand cannot be explained by any intrinsic difference between the hands, but only by their different relationships to space. It is logically possible that there exists a right hand and a left hand which are exactly identical in all their ordinary properties. Nonetheless they are different, and that is because of their relation to space. The point I am going to make about additive and non-additive properties has affinities with Kant's remark.

Graham Nerlich has generalized Kant's insight, and Nerlich regards the property of being a left hand and a right hand as particular cases of the general concept of *enantiomorphic properties*.¹² In order to give a definition of this concept we need the concepts of 'rigid motion' and 'reflective mapping'. A rigid motion is a movement in which the moving thing neither bends nor stretches; a reflective mapping is a function which takes one from the space of a thing to a space where a mirror image of the thing exists. In a three-dimensional space this may mean simply that, in the function describing the thing, the sign of the x-values are changed, i.e. the y-z plane acts as a mirror. Having introduced these concepts Nerlich says:

Now we can express the idea of enantiomorphism in a new way which has nothing to do with possible worlds or with the relation of one hand to another (actual or possible) hand or body. It does, however, *quantify over all mappings* of certain sorts. We can assert the following: Each reflective mapping of a hand differs in its outcome from every rigid motion of it.¹³

The concept of enantiomorphism requires that space is a real unity independently of the contained objects. In such a three-dimensional space there is no rigid motion which can move a hand into a reflective mapping of it. In a two-dimensional space there is no rigid motion which can move an L into an \bar{L} , although there is such a motion in a three-dimensional space. Being an 'L' is an enantiomorphic property in a two-dimensional space but a homomorphic property in a three-dimensional space. Whether a property is enantiomorphic or not depends on the space containing it. If space were four-dimensional, being a left hand would *not* be an enantiomorphic property.

Pre-mathematical addition

Instead of 'rigid motions' and 'reflective mappings' of things we shall look at 'shortenings' of

¹² See G. Nerlich, *The Shape of Space*, Cambridge University Press: London 1976, chapter 2.

¹³ Op. cit. p 35.

'causally rigid' things. In thought we shall investigate what necessarily happens and what necessarily cannot happen with different properties when a thing in which they inhere is shortened. We shall be making *Gedankenexperimente*.¹⁴

Let us assume that we have a body (let's say a rod) with a specific colour hue, a specific volume, a specific mass, a specific shape and a specific density placed along a straight line. Furthermore, assume that no properties of the body will change due to causal processes triggered by a shortening of the body. The body is, we shall say, 'causally rigid'.

If, now, we make a cut orthogonal to the line along which our causally rigid rod is placed, and take away a bit from the rod, then, what happens with the inhering properties? Let us first compare volume and phenomenological colour hue. In our thought experiments we shall regard colours not as electromagnetic radiation but, with common sense, as properties inhering in things, i.e. we shall regard colours as primary qualities like volumes and shapes. The shortened rod will, given our presuppositions, instantiate the same specific colour hue as the original rod, but it will necessarily instantiate another and smaller volume. If we repeat this operation of shortening, then the remaining rod still instantiates the same colour hue but an even smaller volume. Obviously, such an operation can be repeated an arbitrary number of times.

This simple thought experiment shows that determinate¹⁵ colour hues (regarded as primary qualities) and determinate volumes are differently related to space. A colour-determinate can be instantiated in a thing without other colour-determinates being instantiated in the possible spatial parts of the thing, but a volume-determinate cannot possibly be instantiated if other volume-determinates are not instantiated in the possible spatial parts of the thing. There is a part-whole relation between volume-determinates which is lacking in the case of colour-determinates. A volume-determinate *spatially includes* other volume-determinates (those which are of less volume) as parts, while a colour determinate *spatially excludes* other colour-determinates.

It may be argued that the difference just pointed out is, for two reasons, of no ontological importance. First, volume is a special property with a very close relation to spatial extension; second, colour hue is in fact not a primary but a secondary quality. However, there are other properties which embody the distinction I am trying to unfold. Let us now see how the masses and densities of causally rigid rods behave when the rods are shortened.

Assume that we have two compact and homogeneous rods with different masses.

¹⁴ This part of the paper repeats views earlier put forward in chapter 4 of my book *Ontological Investigations*, Routledge: London 1989.

¹⁵ 'Determinate' is here and in what follows used in the sense it has in the philosophical distinction between determinates and determinables. A specific colour hue is a determinate of the determinable colour hue.

Compactness means that the rods contain no void, and homogeneity means that all parts of the rods of the same volume are equal in all respects. If we imagine successively smaller parts of these two rods (the corresponding parts however being equally large), then the mass necessarily diminishes successively in both cases. But for arbitrarily small parts there nevertheless remains the difference that the mass of the one rod is greater than that of the other. This difference is due to the fact that the rods have different densities.

There are two kinds of densities of a thing, real density and nominal density. On the one hand there is the real density of each compact and homogeneous part, on the other there is the mean value density. In homogeneous things the two densities are numerically identical; in heterogeneous things the mean value density always differs from the real density of at least some part. Mean value density has to be conceived nominalistically; it is merely a conceptual construct. This construct, however, gets its content from really existing densities, because if mass is a real property then the densities of homogeneous parts have to be real properties, too. If the densities were not real properties, the real difference between two rods of equal volume but different masses would be inexplicable. A real difference cannot be explained by a conceptual construct.¹⁶ As a tone has to have both pitch and volume, a material body has to have both mass and density. This seems to be a kind of synthetic *a priori* truth.¹⁷

Let us now take a second look at the shortenings of the rods in our thought experiment. Whereas the masses of the rods necessarily are diminished when the rods are shortened, the density of the homogeneous rods necessarily remains the same. In this thought experiment mass behaves like volume in our first experiment and density behaves like colour hue. Mass and density are differently related to space. A mass-determinate *spatially includes* other mass-determinates (those which are of less mass) as parts, while a density-determinate is only itself and *spatially excludes* other density-determinates.

Both mass and density can be linearly ordered in a relation of greater-less and quantified. But while an instance of a greater mass necessarily has potential spatial parts which have less mass, an instance of a larger density has no spatial parts at all with less density. Each density is, like each (phenomenological) colour hue, an indivisible unit which excludes other similar units, i.e. other densities. It should to be noted, though, that densities and colour hues have to be distinguished from density *patterns* and colour *patterns*. If one removes a part of a pattern one usually gets a different pattern, which means that the patterns behave more like volume and mass. Different parts of a thing may of course have different densities, but each such

¹⁶ Armstrong, though, claims that density is a reducible property; see 'Are quantities relations? A reply to Bigelow and Pargetter' in *Philosophical Studies* 54 (1988), pp 305-16, in particular pp 313-15.

¹⁷ For a defence of *a priori* knowledge, see my *Ontological Investigations*, chapter 16.2 'A new way of looking at the synthetic *a priori*'.

density behaves in the way explained.

The properties of volume and mass are such that, in and of themselves, when instantiated in space, they *include* other volume-determinates and mass-determinates, respectively. Therefore, I shall call them *spatially inclusive properties*. A specific (phenomenological) colour hue, on the other hand, *excludes* other colour hues in the space where it is instantiated. Colour hue is, I shall say, a *spatially exclusive property*.¹⁸

The converse operation of shortening is lengthening or *physical addition*, which means that the distinction between inclusive and exclusive properties is of relevance for such addition. Inclusive properties are physically additive whereas exclusive properties are necessarily physically non-additive. When two things are joined to each other, then the inclusive properties of the original thing necessarily become included in a new instance of the same determinables, whereas an exclusive property cannot possibly be included in a new instance of the determinable to which it belongs. If two things are joined to each other then necessarily both a new volume-determinate and a new mass-determinate become instantiated; the new determinate volume and mass are the sums of the old volumes and masses, respectively. However, no new instance of any real density comes into being. Densities cannot because of their 'nature' be physically added.

Both Hempel and Bunge talk of the joining or adding of things on the one hand and mathematical addition on the other. According to my account there is something in between these two operations: there is also physical addition of *properties*. When things are joined, some properties (the inclusive ones) by their 'nature' combine together additively whereas other properties (the exclusive ones) because of their 'nature' cannot do so. This is a fact which is lost from view in the top-down approaches, and this is the fact which makes a realist distinction between additive and non-additive quantities possible.

When an inclusive property is quantified we necessarily get an additive or extensive quantity, and when an exclusive property is quantified we necessarily get a non-additive quantity. Inclusive and exclusive properties differ in their relation to space independently of whether they are quantified or not. Some exclusive properties (e.g. density) are quantifiable whereas other exclusive properties (e.g. phenomenological colour hue) are not. Also, many inclusive properties (e.g. volume and mass) are quantifiable; but not all are such. Shape is an inclusive property which is not quantifiable. Let us see why.

¹⁸ My concept of inclusive property is very similar to Armstrong's concept of *non-relationally structural property*; see Armstrong, *A Theory of Universals. Universals & Scientific Realism vol. II*, Cambridge University Press: Cambridge 1978, pp 68-71. I noted already in my *Ontological Investigations* (see note 14) that probably the concepts have the same *extension*. There are however differences which are mentioned in the book, the most important being that Armstrong does not have a realist view of determinables and does not give space the central function I give it.

Imagine a causally rigid thing with a certain shape, and imagine a straight partition line through it. If we imagine orthogonal cuts and shortenings of the thing, we see that the remaining thing instantiates constantly new shapes (i.e. shape-determinates). Conversely, if we join two things to each other then the original shape-determinates are added and a new shape becomes instantiated. Shapes behave in relation to space just like volumes and masses; shape is an inclusive property. It differs, however, from volume and mass in the following way. The partition line in our thought experiments of shortening (and lengthening) can be drawn in a variety of ways. In the case of quantifiable properties like volume and mass, *every* determinate which can be found along one partition line can also be found somewhere along every other partition line, but this is not the case with shapes. As can easily be seen by drawing some figures, shortenings along one partition line can yield shapes which cannot possibly be found along a certain other partition line. This fact explains why shape is spatially inclusive and additive but is nonetheless not quantifiable and does not give rise to an additive *quantity*.

The approach to physical additivity taken here, with shortenings and lengthenings along a conventionally determined partition line, allows shape to be considered an additive property. If such a line is given, one shape can be conjoined to another, and these two can be added in a well-defined way. They form *one* new shape, not a shape pattern, as would have been obtained if shape were an exclusive and non-additive property. Thus there is physical addition even for non-quantifiable properties, but only for inclusive properties.

The important point which emerges from the discussion above is that there is a kind of addition (property addition) which is independent of the existence of numbers and mathematical addition. This kind of physical addition amounts to more than merely forming an aggregate out of different things. Therefore, I want to call it pre-mathematical addition. Of course, one can speak of the converse operation as pre-mathematical subtraction.

Pre-mathematical vectorial addition

The pre-mathematical addition discussed above relates to *scalar* addition. Now I want to show that there is also a kind of pre-mathematical *vectorial* addition. Thus even some vectors can be construed bottom up.

Vectors differ from scalars in that they have a direction. The concept of space used when pre-mathematical scalar addition was defined was that of an isotropic space, i.e. a space without intrinsic directions. Time, however, is directed, which affords me the clue to the next thesis:

Pre-mathematical vectorial addition is addition in time of temporally inclusive properties,

whereas pre-mathematical scalar addition is addition in space of spatially inclusive properties.

The kind of thought operation by means of which I introduced the distinction between spatially inclusive and exclusive properties can equally well be applied in time. Instead of spatial shortening we then have temporal shortening. In thought we take away parts of a thing's 'life time'. It is, in analogy with spatially inclusive and exclusive properties, possible to distinguish between *temporally* inclusive and exclusive properties.

All the properties so far discussed (volume, color hue, mass, density, and shape) are temporally exclusive properties. Assume that we have a thing which does not change any of these properties during a five minute interval. If we take away the last minute, or any part of the interval you like, this does not affect the thing's properties. These properties do not add in time, and that is because of their 'nature'; they are *exclusive* in time. Therefore, also, their physical dimension need not and does not contain any reference to time. This, by the way, seems to be true for most of the properties referred to by modern physics.

What is true for a temporally exclusive property is, however, not true for a *change* of the same property. Changes of temporally exclusive properties constitutes *patterns in time*. Ordinary patterns, i.e. spatial patterns, are inclusive in space and temporal patterns are inclusive in time. If you think away part of the change, what remains is *another* change which was part of the larger change.¹⁹

Movement, or change of place, is a temporally inclusive property. If one thinks away one part of a movement then, necessarily, another and shorter movement remains. A spatially inclusive property is necessarily extended in space, but movement is necessarily extended in time. There can be no movement (speed or velocity are other properties, see below) or change of place in a point of time. This is true even mathematically. When you integrate a velocity function in order to get a particular movement between t_1 and t_2 , you get the result zero if t_2 equals t_1 .

The similarity between spatial and temporal inclusiveness goes even further. Movements can, pre-mathematically, be added in time just like spatially inclusive properties can be added in space. Two movements which are joined in time constitute a new movement which is the sum of the two included movements. Also, speed is to movement (in time) what density is to mass (in space).

Just as two things may be of equal volume but differ in mass, so two movements can be of equal length but differ in time. In the former case the difference shows the existence of density or intensity of mass; in the latter case the difference shows the existence of speed or

¹⁹ Cf. *Ontological Investigations*, chapter 6.1.

intensity of movement. If we look at a homogeneous (i.e. uniform) motion, we realize immediately that speed is temporally exclusive. We can shorten the movement we are considering how many times we want without, thereby, affecting the speed of the movement. This means that, in contradistinction to movements, speeds cannot possibly be added in time.

Just as mass and density are differently related to space so movement and speed are *differently related to time*.

A rod and a movement may be equally long, but the length of the rod lacks direction whereas the length of the movement has direction. A movement is *from* a place *to* a place. True, an ordinary *measurement* of the length of a rod has a direction, but that is not because the rod has a direction but because the measurement is extended in time. First you look at one end of the metre rod and then you look at the other. The one who makes the measurement goes ("moves") *from* one end *to* the other end. The direction is not in the rod itself as there is direction in the movements themselves.

Movements can be added in time independently of their mathematical representation, and when they are so added they are added with their direction. Time-extended motion is inclusive in time and quantifiable. Therefore, it can be added vectorially both physically and mathematically.

Concluding remarks

I have argued that some quantities are such that their additivity has to be understood realistically or 'bottom up'. That is, some predicates refer to properties which exist *in re* and which, by their 'nature', are such that they are additive. In these cases, we can add only because the properties are possible to add - either spatially or temporally. I have put 'nature' within scare quotes because this nature is not something internal to the properties; it is a relationship between the properties in question and space (or time). Even simple properties can be differently related to space and time.

Now, of course, the fact that some quantities and their additivity has to be understood realistically does not mean that no quantities should be understood nominalistically. With regard to modern physics, with its heavy reliance on spatially and temporally punctual quantities, a really interesting question is whether any of the postulated point-magnitudes should be given a realist interpretation. What about, for instance, density in a zero-dimensional point and velocity in a momentary point of time? Can there be density without mass and velocity without movement? These questions, however, will not be dealt with here. I want to end by putting them forward because I think that, with respect to point-magnitudes,

too, many philosophers too easily take a nominalist stand as self-evident.²⁰

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²⁰ For attempts at realist interpretations see my *Ontological Investigations* chapters 6.2, 6.3, and 7.2 and J. Bigelow and R. Pargetter, 'Vectors and Change' in *The British Journal for the Philosophy of Science* 40 (1989), pp 289-306.