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Functions and Shapes in the Light of the International System of Units

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Abstract

Famously, Galilei made the ontological claim that the book of nature is written in the language of mathematics. Probably, if only implicitly, most contemporary natural scientists share his view. This paper, in contradistinction, argues that *nature is only partly written in the language of mathematics*; partly it is written in the language of functions and partly in a very simple purely qualitative language, too. During the argumentation, three more specific but in themselves interesting theses are put forward: first (in section 3: ‘Shape as a purely qualitative physical dimension’), *there are more shapes than real numbers*; second, (in section 4: ‘The four-partite structure of amount of substance quantities’), *the metrological notion ‘amount of substance’ can profitably be exchanged for ‘number of entities’*; third (in section 5: ‘Functioning as a purely qualitative physical dimension with degrees’), *prototypical concepts will always be scientifically important*.

Keywords: quantities, functions, shapes, SI system

1. The International System of Units (SI)

The International System of Units (SI), created in 1960, has been for several decades the authoritative system of units within the natural sciences and the life sciences. Since 1971, when the 'mole' was added, it contains seven base quantities to which many other fundamental quantities in physics, chemistry, biology, and medicine can be related by means of multiplication, division, or both, e.g., to take some very basic dimensions, (area) = $(\text{length})^2$, (velocity) = $(\text{length}) \cdot (\text{time})^{-1}$, and (energy) = $(\text{mass}) \cdot (\text{length})^2 \cdot (\text{time})^{-2}$'. The seven quantities are (SI 2006, p. 116):

Base Quantity (Dimension)	***SI Base Unit***	***Symbol***
length	meter	m
mass	kilogram	kg
time, duration	second	s
electric current	ampere	A
thermodynamic temperature	Kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

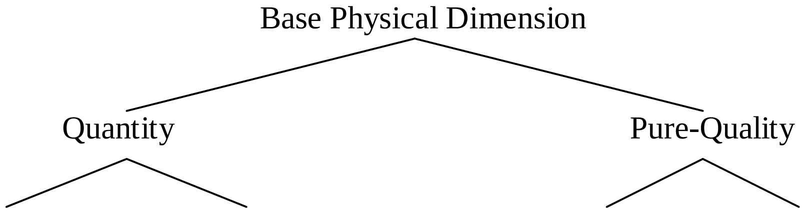
Table 1: *The base quantities of SI.*

SI aims at standardizing measuring units. Therefore, by definition one might say, it is concerned only with quantified physical properties. Famously, Galilei claimed that the book of nature is written in the language of mathematics. If he is right, and all the properties of relevance for the natural sciences and the life sciences can be quantified, the seven base quantities of SI could be regarded as the alphabet of nature, but I will argue that he is wrong. In particular, I will show that shapes, functions, and functionings cannot be quantified, even though they are, and should be, investigated by both the natural sciences and the life sciences. This fact notwithstanding, they have interesting metrological features in common with some features exhibited by the base quantities of SI. The views I will be arguing for can also be stated thus:

(a) some properties that are important for the natural sciences and the life sciences are purely qualitative

(b) quantitative and purely qualitative properties have some metrologically important features in common.

If I am right, physical quantities and qualities can be classified as in Table 2 below, where the SI dimensions take account only of the left half of the table. In the table, both the ordinary term ‘property’ and a more general but similar term are needed; and for various reasons I have chosen ‘physical dimension’. Note that there is in science no general taboo on talking about dimensions that are not quantified. For instance, it is often said that social institutions possess the three *dimensions* of structure, function, and culture. Since, as I will make clear, quantities necessarily contain a kind of quality (physical dimension), the contrary opposite of ‘quantity’ is called ‘pure quality’. Why the table gives ‘amount of substance’ a column of its own, in what way there can be degrees of a ‘pure-quality’, why there is a distinction between ‘function’ and ‘functioning’, and why some other details look the way they do will be explained in due course.



Property:	Amount of substance:	Property:	Degree of functioning:
length	of elementary entity E_1	2D-shape	of function F_1
mass	of elementary entity E_2	3D-shape	of function F_2
time	of elementary entity E_3	pattern	of function F_3
electric current	…	function	…
temperature	…		…
luminous int.	of elementary entity E_n		of function F_n

Table 2: *Proposed overarching classification of base physical dimensions.*

SI contains a distinction between quantities of the *same dimension* and quantities of the *same kind* that is not part of either Table 1 or Table 2. For instance, energy, work, and moment of force have the same dimension, (mass) · $(\text{length})^2$ · $(\text{time})^{-2}$, but are regarded as being different kinds of quantities. Necessarily, if two quantities are of the same kind they have the same dimension, but not vice versa. Since it has been argued that the expression ‘same dimension’ is not needed and even misleading (Emerson 2005), I want to declare that, if necessary, the essence of my views can be re-stated by substituting ‘same kind’ for ‘same dimension’.

2. The tri-partite structure of property quantities

In the latest edition of *International Vocabulary of Metrology – Basic and General Concepts and Associated Terms (VIM)*, the concept of ‘quantity’ is defined as follows (VIM 2006, def. 1.1, p. 16, italics added):

- quantity $=_{\text{def}}$ property of a phenomenon, body, or substance, to which a *number* can be assigned with respect to a *reference*.

When the base quantities are used to report results of measurements, they refer to properties, but these properties are not directly referred to. The referring takes place via something; in VIM called a ‘reference’. In relation to the base quantities, this reference is the base unit chosen. When the units and symbols for the base quantities of SI are not as in Table 1 just listed, but used to report results of measurements, then of course a numeral referring to a mathematical number has to be combined with the symbol of the base unit in question. Examples of quantity *statements* are ‘B is 5.03 m long’ and ‘B has a mass of 1.42 kg’; the forms of the corresponding two quantity *expressions* are ‘x m length’ and ‘x kg mass’. As is clear from Table 1, ‘each of the seven base quantities used in the SI is regarded as having its own dimension (SI 2006, p. 116)’. Such dimensions (length, mass, etc.) are independent of the base unit in question, since other base units (such as foot for length and pound for mass) could have been chosen.

Expressions such as ‘5.03 m’ and ‘1.42 kg’ have only two parts, and quantity expressions are in metrological contexts mostly described as consisting of exactly two parts, a quantity value and a unit (Dybkaer 2004b, p. 35). But this description is due to the fact that the base unit is taken for granted. For the purposes of this paper, however, it is important to make explicit the fact that the most general form of quantity expressions (Q) contains three parts:

- Q = (quantity value) · (base unit) · (physical dimension).

Apart from the pragmatically chosen base unit, there is necessarily both a mathematical number (quantity value) and a physical dimension. In VIM’s quantity definition, this tri-partite distinction is only partially displayed by the terms ‘number’ and ‘reference’. Later, however, the concepts of ‘quantity dimension’ and ‘quantity value’ are clearly distinguished from the base unit (VIM 2006, def. 1.7, p. 22, and def. 1.19, p. 32, respectively). The following holds true independently of the base unit chosen:

- necessarily, each base *quantity* brings in both a *quantity dimension* and a *quantity value*.

This dimension-value duality of quantities is not, unlike the base unit, something that pragmatic considerations may influence. The necessity just stated can be re-phrased in terms of ‘identity-in-difference’. In order for quantity talk to make sense, there has to be something that can be *identical* (the dimension) *in* many *different* determinate properties (values). Furthermore, these determinate properties of the dimension in question have to be such that they can be linearly ordered in the logical sense of ‘linear order’ (not to be conflated with ‘linear equation’). We do not directly in nature meet any quantities, but this does not mean that there is nothing in nature that corresponds to quantities. We create the linear orderings that underlie the base quantities, but this is possible only because these physical dimensions and their determinate properties have – in and of themselves – a certain character. Let me anticipate my point about shape by means of an analogy: as nature makes it possible for us to build walls with stones but not with water, nature makes it possible for us to order linearly each of the base SI dimensions but not shape.

In order to obtain quantities, more than a linear ordering is necessary. Before a base unit can be chosen and quantities talked about, relations between intervals in the linear ordering at hand have to be found; for instance, the distance between 5 and 7 kg should be exactly the same as the distance between 1021 and 1023 kg. However, neither this requirement nor the distinction between interval scales and ratio scales (Kaplan 1964, §22; Hand 2004, Chapter 2.2.5) are of any special importance for the discussion in this paper; and they will be mentioned only in passing.

When quantity statements are used to describe or name a spatiotemporal instance of a determinate property, they implicitly relate this determinate instance to all other possible instances of determinate properties of the same physical dimension. When it is said ‘B_1 has a mass of 1.42 kg’, the following kinds of states of affairs are implied (point c would not follow if mass was not represented by a ratio scale):

(a) B_1 is ascribed a determinate *monadic* property

(b) if also B_2 is said to be 1.42 kg, then – implicitly – B_1 is ascribed the *relational* property of ‘having the same mass as B_2’

(c) if B_3 is said to be 2.84 kg, then – implicitly – B_1 is ascribed the *relational* property of 'having twice the mass of B_3'.

Much of the usefulness of quantity talk comes from all these handy implicit ascriptions of relational properties of similarity and distinct dissimilarity. Necessarily, similarity and dissimilarity are similarity and dissimilarity in a certain respect; and it is this 'respect' that is made explicit by the quantity dimension in question.

If we view a situation as it looks *before* any quantification at all is made, then the following holds true:

A. if we cannot distinguish between a specific dimension (length, mass, etc.) and different determinate properties of this dimension, then we cannot start to think about quantifying

B. if we have found a dimension with many determinate properties, but the set of these determinate properties cannot be linearly ordered (as in the case of shape), then no quantification is possible

C. if such a linear ordering (ordinal scale) is possible, but there is no way of identifying equally long distances in it, then quantification is nonetheless impossible

D. if also such distances can be identified, then it is possible to start to discuss what base unit to choose in order to create practically useful quantities out of the order found (this discussion is intimately connected with a discussion around the possibility of obtaining not only an interval scale but also a ratio scale; it can also lead to a discussion whether the scale for practical reasons should be turned into a logarithmic scale).

The conventionality of a base unit is a conventionality only against the background of some hard facts about a physical dimension and its determinate properties (values).

3. Shape as a purely qualitative physical dimension

Before answering – in the negative – the question whether shape is a quantifiable physical dimension, I will say some words about what is normally meant by 'shape', and the way shapes are important in the natural sciences. (At the end of the section, I will make some brief remarks also on the possibility of quantifying patterns and functions.)

By shape is in everyday life mostly meant the two- and three-dimensional outlines of surfaces and things. Such shapes are *closed* shapes, but in science there is often talk also of *open* shapes, e.g., parabolas and hyperbolas. In what follows, the extension of the term ‘shape’ includes both open and closed shapes; and not only geometrical or fractal such shapes, but any arbitrary shape whatsoever. If a shape (at a certain place in space) by means of translation, rotation and uniform scaling can be made congruent with a shape at another place, then these two shapes are two instances of the same (kind of) shape. In other words, shapes are invariant to location, rotation, and size.

Shape so delineated is a property dimension, which is quite consistent with the fact that some shapes in turn have (second-order) properties such as ‘being symmetric’, ‘being regular’, ‘being polyhedrical’, and as having mathematical properties such as ‘eccentricity’. In some contexts, words such as ‘form’ and ‘figure’ are used instead of ‘shape’.

Ever since the birth of modern science, shapes have been of considerable significance in several disciplines. For instance, in astronomy the shape of the orbits of planets and asteroids has always been of central interest; and in classifications of crystals, plants and animals, shape has always been one among the features used. The fact that the DNA molecule has the shape of a double helix is crucial when it comes to understanding the way it functions. When protein macromolecules perform their biological functions, the proteins fold into specific spatial configurations, and in order to understand these processes (caused by various bondings) it is necessary to determine the three-dimensional structure of the proteins. Structural biology has even become a special branch of molecular biology. In short, the natural sciences and the life sciences are using, and have always been using, the concept of shape without quantifying it in the way the SI dimensions are quantified. Therefore, shape has in fact long deserved at least a temporary place in a table like Table 2. My argument below intends to show that this place cannot be considered accidental.

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That shape is a physical dimension with many qualitatively different determinate shapes is beyond all doubt. In ordinary language, determinate shapes are represented by means of words such as ‘round’, ‘elliptical’, ‘triangular’, and ‘star-shaped’; but we can also represent determinate shapes by means of pictures. This means that condition A (see end of Section 2) for a quantification of shape is fulfilled, but, I will argue, condition B (a linear ordering) cannot be fulfilled. Therefore, shape should be regarded as an important but nonetheless in principle purely qualitative physical dimension that has many qualitative values. I have here and in what follows exchanged the old philosophical terms ‘determinable’ and ‘determinates’ for the metrological and more modern terms ‘dimension’ and ‘quality value’; compare Johansson (2002). Semantically, we should keep the tri-partite structure of quantity expressions (Q) such as ‘1.42 kg mass’ distinct from the two-partite structure of pure quality expressions (QL) such as ‘round shape’:

- Q = (quantity value) · (base unit) · (physical dimension)
- QL = (quality value) · (physical dimension).

The quantifications of length, mass, etc. are quantifications of *all* the *possible* determinate properties of the physical dimension in question; there are no possible determinate mass properties outside of the quantity dimension mass. Therefore, the problem to be dealt with is whether or not *all possible* shapes can be quantified. It is well known that at least one subset of shapes, the ellipses, can be linearly ordered. This is done by means of their eccentricity (e). If the length of the semimajor axis of the ellipse is called a and the length of the semiminor axis b, then each and every ellipse can be given an eccentricity value larger than zero but lesser than one according to the formula: $e = \sqrt{\ } (1 - b^2/a^2)$. Such orderings can be very useful and important for specific purposes in science, but this is beside the quantification problem now at issue. What has been done with respect to ellipses cannot, I claim, be done for all shapes. Whereas it makes good sense to say ‘this ellipse has an eccentricity of 0.73’, it is impossible to give every shape a number related to a base unit (e.g., ‘morphe’), and make sense of sentences such as ‘this shape has a morphe of 2.31’. In my attempted non-quantifiability proof, I will only discuss 2D-shapes; if they cannot be quantified neither can the more complex 3D-shapes.

In a quantification, the determinate properties of a physical dimension are given a one-to-one mapping onto the real numbers. It is an obvious truth that if a domain S has a larger cardinality than another domain R, then there can be no one-to-one mapping between S and R. This means that if there are a larger number of 2D-shapes than there are real numbers, then 2D-shapes cannot possibly be quantified; and this is what I will show is the case.

The first thing to be noted is that since every 2D-shape has some finite extension in space, it can always be divided into a number of segments. Furthermore, since the shape/line in question constitutes a continuum, it must contain at least as many points or infinitesimally small segments as there are real numbers.

The second thing to be noted is that there are different kinds of shape segments. Using ideas from Hoffman and Richards (1982), we can to begin with state that there are five basic different kinds of such segments:

1. curving-in (⊃)
2. curving-out (⊂)
3. angle-inward (>)
4. straight (–)
5. angle-outward (<).

Each infinitesimal segment of a two-dimensional shape can be ascribed one and only one of these five different kinds of open shapes. Even though the curving-in by rotation can be turned into the curving-out (and the angle-inward into the angle-outward), and they are in this sense identical shapes, they are nonetheless *as shape segments* different. If a curving-in is joined to another curving-in of the same kind, one obtains another shape than if it is joined to a corresponding curving-out. But more can be said about the segments. Importantly, there are as many different curving-ins as there are real numbers. This can be shown as follows.

First, let us look at the curving-ins at the vertex of hyperbolas. Like ellipses, hyperbolas can be linearly ordered by means of an eccentricity measure, and each different eccentricity implies a different curving-in at the vertex. According to the usual eccentricity formula for hyperbolas, eccentricity can vary from >1 to infinity, which means that there are at least as many curving-ins as there are real numbers between 1 and infinity. Second, parabolas have another curvature at the vertex than the hyperbolas; let us associate all these parabola curvatures with the number 1, since they have the same eccentricity, 1. Third, let us look at the curvatures at the end points of the major axis of the ellipses. They differ from that of the hyperbolas and parabolas, and they also differ from each other depending on the eccentricity of the ellipse. This eccentricity varies from zero (the circle) to <1. Adding the curving-ins of hyperbolas, parabolas, and ellipses together, we will obtain at least as many different curving-ins as there are real numbers. And there are of course as many curving-outs as there are curving-ins.

Furthermore, the angle-inwards and the angle-outwards can together with the straight line be linearly ordered according to ‘degree of angularity’, and be given values between 0 and 360. One then ascribes angle-inward degrees that are larger than zero but smaller than 180^{o}, gives the straight line the value 180^o, and angle-outward degrees larger than 180^o but smaller than 360^o.

There can be no doubt that there are at least as many different kinds of shape segments as there are real numbers. In fact, since each shape segment is an open shape of its own, we can at once see – not astonishingly – that there are at least as many possible open shapes as there are real numbers. The task is, however, *to prove that there are more possible shapes than real numbers*. In order to do this, some transfinite mathematics is needed. Two things should be remembered. First, the set of real numbers have the cardinality *aleph-one* ($\aleph_1$), which is larger than the cardinality of the set of natural numbers, *aleph-null* ($\aleph_0$). Second, raising 2 to the power of a cardinal – be it finite or infinite – always gives a larger cardinal, i.e., $2^\aleph > \aleph$ (see e.g., Moore 1990, Chapter 10.3).

Let us now take a look at what 2D-shapes (lines) may look like when analyzed into a number of segments that might be shape segments of different kinds. Shapes that consist of two segments chosen among two different kinds of segments (a, b) can, following ordinary combinatorial reasoning, be combined in four different ways. But there are nonetheless not four, but three different shapes. This is due to the fact that shape is invariant to rotation. If we represent the four possible combinations as <a, a>, <b, b>, <a, b>, and <b, a>, then the last two combinations can by rotation be made identical. Similarly, if we look at two particular segments chosen among *three* different kinds of segments (a, b, c), then we obtain nine combinations, namely <a, a>, <b, b>, <c, c>, <a, b>, <b, a>, <a, c>, <c, a>, <b, c>, <c, b>, but the last three pairs represent for reasons of symmetry only three different shapes. Since by rotation only symmetrical combinations can be made identical, there will always in this kind of combinatorial enterprise be more shapes than the total number of possible combinations divided by two. In the case of 2 segments chosen among 2 kinds-of-segments, there are more than $2^2/2$ shapes (namely 3); in the case of 2 segments chosen among 3 kinds-of-segments there are more than $3^2/2$ shapes (namely 6); 4 kinds-of-segments give $4^2/2$ shapes; and so on. If we look at shapes consisting of three segments, we obtain, for two, three, and four different kinds-of-segments more than $2^3/2$, $3^3/2$, and $4^3/2$ shapes, respectively. In general, out of a finite number of segments, m, chosen among n different kinds-of-segments, more than $n^m/2$ different possible shapes can be produced. Letting n and m move into infinity does not change anything in the remarks made.

From what has been said the following ensues. If we fix the number of segments to 2, and then let the number of kinds-of-segments progress from 2, 3, and 4 to an arbitrary finite number n, and further into the infinite cardinal numbers $\aleph_0$ and $\aleph_1$, it holds true that we obtain, in turn, at least the following number of shapes: $2^2/2$, $3^2/2$, $4^2/2$, $n^2/2$, $\aleph_0{}^2/2$, and $\aleph_1{}^2/2$ (this progression makes up the second *line* in Table 3). However, since $\aleph_0{}^2/2 = \aleph_0{}^2 = \aleph_0$ and $\aleph_1{}^2/2 = \aleph_1{}^2 = \aleph_1$, we do not here find more shapes than real numbers. Let us try another course (which constitutes the second *column* in Table 3).

Now, instead, we fix the number of kinds-of-segments to 2, and then let the number of segments progress from 2, 3, and 4 to an arbitrary finite number m, and further into the infinite cardinal numbers $\aleph_0$ and $\aleph_1$. With respect to the finite numbers there are at least, in turn, the following numbers of shapes: $2^2/2$, $2^3/2$, $2^4/2$, and $2^m/2$. Aleph-null number of segments gives us $2^{\aleph 0}/2$ number of shapes. Since $2^{\aleph 0}/2 = 2^{\aleph 0} > \aleph_0$, we have found a set of shapes with a cardinal number larger than that of the natural numbers (probably, it is equal to $\aleph_1$), but we have not yet found a set of shapes whose cardinal number is provably larger than the set of real numbers. However, aleph-one number of segments supplies us with such a set. Even if there are only 2 kinds-of-segments possible, but there are $\aleph_1$ number of segments, there are more than $2^{\aleph 1}/2 = 2^{\aleph 1} > \aleph_1$ number of shapes.

If we do not keep the two progressions presented distinct, we can argue as follows:

(i) any 2D-shape can be divided into $\aleph_1$ segments (follows from the facts that a 2D-shape is an extended continuum in space and that every continuum contains at least as many parts as there are real numbers)

(ii) there are at least $\aleph_1$ different kinds-of-segments (follows from the analysis of different kinds of shape segments made)

(iii) there are at least $\aleph_1{}^{\aleph 1}$ different combinations of segments and kinds-of-segments (follows from the fact that each of the $\aleph_1$ segments can be chosen among $\aleph_1$ kinds-of-segments)

(iv) there are at least $\aleph_1{}^{\aleph 1}/2$ different 2D-shapes (follows from iii and reflections on the fact that shapes are invariant to rotation)

(v) since $\aleph_1{}^{\aleph 1}/2 = \aleph_1{}^{\aleph 1} \geq 2^{\aleph 1} > \aleph_1$, *there are more 2D-shapes than real numbers*

(vi) Q.E.D.

Table 3 below makes the double progression discussed directly visible.

	2 kinds:	3 kinds:	4 kinds:	n kinds:	$\aleph_0$ kinds:	$\aleph_1$ kinds:
2 seg.:	$2^2/2$	$3^2/2$	$4^2/2$	$n^2/2$	$\aleph_0{}^2/2$	$\aleph_1{}^2/2 = \aleph_1{}^2 = \aleph_1$
3 seg.:	$2^3/2$	$3^3/2$	$4^3/2$	$n^3/2$	$\aleph_0{}^3/2$	$\aleph_1{}^3/2 = \aleph_1{}^3 = \aleph_1$
4 seg.:	$2^4/2$	$3^4/2$	$4^4/2$	$n^4/2$	$\aleph_0{}^4/2$	$\aleph_1{}^4/2 = \aleph_1{}^4 = \aleph_1$
m seg.:	$2^m/2$	$3^m/2$	$4^m/2$	$n^m/2$	$\aleph_0{}^m/2$	$\aleph_1{}^m/2 = \aleph_1{}^m = \aleph_1$
$\aleph_0$ seg.:	$2^{\aleph 0}/2 > \aleph_0$	$3^{\aleph 0}/2 > \aleph_0$	$4^{\aleph 0}/2 > \aleph_0$	$n^{\aleph 0}/2 > \aleph_0$	$\aleph_0{}^{\aleph 0}/2 > \aleph_0$	$\aleph_1{}^{\aleph 0}/2 = \aleph_1{}^{\aleph 0} = \aleph_1$
$\aleph_1$ seg.:	$2^{\aleph 1}/2 > \aleph_1$	$3^{\aleph 1}/2 > \aleph_1$	$4^{\aleph 1}/2 > \aleph_1$	$n^{\aleph 1}/2 > \aleph_1$	$\aleph_0{}^{\aleph 1}/2 > \aleph_1$	$\aleph_1{}^{\aleph 1}/2 = \aleph_1{}^{\aleph 1} > \aleph_1$

Table 3: *Number of possible combinations of number of shape segments and number of kinds-of-segments. The table shows how the number of possible shapes increases as the number of segments of the shape (see the different lines) and the number of kinds-of-segments (see the different columns) increase. In the lowest line, but nowhere else, there are more different shapes than real numbers.*

As now put forward, this non-quantifiability proof applies to 2D-shapes in general, and by implication to 3D-shapes. It might be claimed, however, that just as length and area are different dimensions in the SI, open shapes and closed shapes should for some topological reasons be regarded as two different dimensions, too. Let us first look at open shapes. Every combination of only one curving-in segment with one curving-out segment results in an open shape. Since there are both at least as many curving-ins and as many curving-outs as there are real numbers, the proof put forward goes through even when constrained to open shapes only.

What then to conclude about closed shapes? In order to deliver only such shapes, the combinatorial enterprise has to be qualified by restrictions that ensure that the first segment of a certain shape becomes connected to the last one. One sufficient such restriction might be that each allowable segment combination has more or stronger curving-ins and angle-inwards than curving-outs and angle-outwards. But since the number of curving-ins on its own equals the real numbers, even the number of closed 2D-shapes is probably larger than aleph-one. Therefore, I claim:

- since neither the set of all possible determinate open shapes, nor that of closed shapes, can be linearly ordered, not only does shape in general have to be considered a non-quantifiable physical dimension; the same holds true for the sub-dimensions of ‘open shape’ and ‘closed shape’, too.

This view does not imply, as I have already said, that it is impossible to quantify smaller subsets of shapes. Neither, does it imply that what is true of shapes cannot be affected by natural laws that are stated by means of quantified physical dimensions. For instance, it is an obvious fact that what the special theory of relativity says about length contraction has implications for what is true about shapes. Just one example: necessarily, what is a circle in one specific inertial frame is a non-circle in all other inertial frames.

At last, a word concerning fallibility. The proof put forward relies on the assumption that traditional combinatorial reasoning about discrete entities can be used even in relation to combinations of infinitesimally small shape segments. If this (or some other) assumption can be shown to be false, such a discovery would nonetheless not in itself show that shapes can be quantified. Still, a positive proof to the effect that there cannot be more shapes than real numbers would be needed; according to my experience, this is mostly simply taken for granted. Furthermore, even if it can be proved that the set of shapes and the set of real numbers have the same cardinality, it also needs to be shown that quantification is substantially possible. This means two things: (i) showing that the shape determinates is of such a character that they can be linearly ordered, and (ii) that in such an order it is also possible to define a measure for what should be regarded as equal distances between two shape determinates.

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Are there other non-quantifiable physical dimensions? Well, if there are other physical dimensions in which shapes are essential parts, then these dimensions must by implication be purely qualitative dimensions, too. And I think there is at least one such dimension: *spatial pattern*. Spatial patterns necessarily involve shapes, but patterns cannot be reduced to shapes. Think for instance of a color pattern. Trivially, it contains colors. But if the different determinate colors were not confined within areas with a certain shape, there would be no color *pattern*. The true components of a color pattern are colors-with-shapes (Johansson 1998). Similarly, if an energy density pattern is de-composed into components, then such components have to be instances of a specific density confined within an area with a certain shape.

In Section 5 I will, among other things, present a distinction between functions and functionings. Here, I only want to state that I find it obvious that if spatial pattern is a non-quantifiable dimension, then the same goes for function taken as a general dimension. Typical functions of the organs of the body are: to store (something), to pump, to protect, to produce, to open, to close, to absorb, and to expel. To line up only this group in a linear order seems to me impossible.

A physical dimension that cannot be quantified cannot be ascribed a base unit. Therefore, Table 1 cannot simply be extended with 2D-shapes, 3D-shapes, pattern, and function; instead a corresponding schema would look like Table 4.

Base Pure-Quality	***Base Unit***	***Symbol***
2D-shape	---	---
3D-shape	---	---
spatial pattern	---	---
function	---	---

Table 4: *Pure qualities cannot be ascribed a base unit.*

4. The four-partite structure of amount of substance quantities

The connected notions of ‘amount of substance’ and ‘mole’ have a more complex structure than the other base notions of the SI, and they have encountered problems that the other notions have not. At first (1960), ‘amount of substance’ and ‘mole’ were not even made part of the system, and after their introduction (1971) papers had to be written that explained that amount of substance was neither the same as mass nor as atomic or molecular weight (McGlashan 1977). Nonetheless, the mole has never have become popular outside a limited range of disciplines such as physical chemistry and chemical thermodynamics. Some metrologists blame this fact on the very term ‘mole’ chosen; M. McGlashan (1994/95) wants it exchanged for ‘ment’ (‘amount of entity’) and R. Dybkaer (2000) for ‘chemon’ (‘chemical amount’). Also, two misunderstandings of the mole seem to arise quite easily: either that the mole is only a mathematical number, or that it has the dimension ‘dimension one’. At the end of this section, I will make clear the rationale behind these misunderstandings, but first I will do what is central to this section: contrast the structure of quantity expressions for amount of substance with that of quantity expressions for properties. If the former structure is not seen, functioning talk in the life sciences cannot be correctly related to the SI (see next section).

The official introduction of the mole in the SI brochure says as follows (SI 2006, p. 115):

> 1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is “mol.”
> 2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

No other introduction of a base unit contains two paragraphs; only one that corresponds to point 1 above. Length, for instance, is introduced as follows:

- The meter is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

Quantity expressions for length have in the SI the general tri-partite form: ‘x · meter · length’, and hadn’t it been for the amendment in point 2, then quantity expressions for amount of substance would, similarly, have had a tri-partite form, namely ‘x · mole · amount of substance’. However, the quantity specification (S) required by the amendment entails that quantity expressions for moles have a four-partite structure:

- QS = x · mole · (amount of substance) · (of elementary entity E)

or, more abstractly:

- QS = (quantity value) · (base unit) · (physical dimension) · (specification).

The expression ‘amount of substance of elementary entity E’ (e.g., ‘amount of substance of HgCl’) is a bit awkward, and it is officially stated that it can be abbreviated into ‘amount of elementary entity E’ (‘amount of HgCl’) (SI 2006, p. 115). This means that the whole four-partite expression ‘x mole amount-of-substance of-E’ is turned into the tri-partite expression ‘x mole of-E’. On the surface, the last expression has the same structure as ‘x meter long’, but in order to obtain the adequate comparison, ‘x mole of-E’ should be contrasted with the two-partite expression ‘x meter’, i.e., contrasted with an expression where, likewise, the physical dimension in question is abbreviated away. And then the fact I want to highlight (and need for Section 5) becomes clear again:

- mole quantity expressions contain one more part than the other base quantity expressions do.

The metrological problems that are discussed in the rest of this section are not necessary to read for the overall purpose of this paper, but what is said shows that the SI not only in fact contains, but probably has to contain, a base dimension whose quantity expressions have a four-partite structure.

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Notwithstanding some impressions to the contrary, the SI does not contradict the contemporary wisdom that scientific conceptualizations, like all others, contain some semantic holism. The fact that there are no logical relationships between the seven base dimensions does not imply that each *concept* of a base dimension can be *understood* independently of the others. It does not even imply that all the seven base *units* are *defined* independently of each other. As can be seen from the introduction of the length unit (quoted above), even though length and time are logically independent base dimensions, the introduction of the base unit ‘meter’ contains an explicit reference to the base unit ‘second’ and its base dimension ‘time’. Similarly, the introduction of the mole contains a reference to the base unit ‘kilogram’ and, thereby, implicitly to the base dimension ‘mass’. This fact is stressed in the SI (2006, p. 111).

Furthermore, but not stressed in the SI brochure, there is sometimes also a dependence between concepts for base dimensions and concepts for derived dimensions. Quite clearly, the base unit ‘meter’ is introduced on the assumption that the velocity of light in vacuum is constant, but velocity is a derived quantity. That is, the introduction of the base unit meter presupposes an understanding of a concept of a derived dimension. In the introduction of the mole, this kind of semantic dependence becomes even wider. It contains the expression ‘as many elementary entities as’, which presupposes the concept of ‘number of entities’, but neither the former nor the latter is referring to any dimension in the SI. However, attempts have been made to enlarge the SI in this direction.

The metrologist I.M. Mills has argued (1994/95) that there is a need to add the number 1 as base unit, and the metrologist, René Dybkaer has supported and expanded the proposal (2004a) by claiming that the corresponding new eighth base dimension should be ‘number of entities’. If this proposal would be accepted, then the second line of Table 5 below should have to be added to Table 1. Since, on this proposal, ‘one’ is a base unit and ‘1’ only a symbol for this unit, it becomes important not to conflate the SI ‘one’ and the SI ‘1’ with the mathematical number 1.

Base Quantity (Dimension)	***SI Base Unit***	***Symbol***
number of entities	one	1

Table 5: *Dybkaer's proposal.*

Since base quantities (e.g., '5.03 m' and '1.42 kg') are unities of a mathematical number and a base unit, a base unit cannot possibly be identical with a mathematical number. In the formula 'Q = (quantity value) · (base unit) · (physical dimension)', only the quantity value can be a mathematical number. Mills' and Dybkaer's proposals make the context free word 'one' and the context free numeral '1' ambiguous between referring to the SI unit and the mathematical number.

Even though Mills' and Dybkaer's proposals have not been fully accepted, their views have gained an implicit recognition. In the latest editions of the SI brochure and VIM we can read (italics added):

> There are also some quantities that *cannot be described in terms of the seven base quantities of the SI at all, but have the nature of a count*. Examples are number of molecules […] Such counting quantities are also usually regarded as dimensionless quantities, or quantities of dimension one, with the unit one, 1 (SI 2006, p. 105-6).

> All of these counting quantities are also described as being dimensionless, or of dimension one, and are taken to have the SI unit one, although the unit of counting quantities cannot be described as a derived unit expressed in terms of the base units of the SI. *For such quantities, the unit one may instead be regarded as a further base unit* (SI 2006, p. 120).

> For *number of entities*, the number one, symbol 1, can be regarded as a *base unit* in any system of units (VIM 2006, p. 26).

In order to understand these quotations, one has to know that there are *derived* quantities that have the base unit one, and are said to be of 'dimension one'. Dybkaer wants a new and eighth base quantity that has the new *base* dimension 'number of entities', but has the same base unit, one, as the traditionally *derived* dimension 'dimension one'.

Dimension one and its base unit one come into being every time a dimension is created by dividing a certain quantity dimension by itself. As the dimension ‘area’ is construed by multiplication of length with itself, $(\text{length})^2$ = (area), the dimension one can be construed by dividing length with itself: $(\text{length}) \cdot (\text{length})^{-1}$ = (dimension one). The base unit for area becomes m^2, and the base unit for dimension one becomes: $m \cdot m^{-1} = 1$. This dimension can, for instance, be used to characterize degrees of plane angles, and it is then called ‘radian’, i.e., (radian) = (*length* of circle arc) · $(\textit{length} \text{ of radius})^{-1}$. The SI contains many other such quantities, e.g., molecular weight, whose dimension is: (dimension one) = $(\text{mass}) \cdot (\text{mass})^{-1}$. The point behind the three quotations from the SI and VIM is that ‘number of entities’ cannot be derived as a ratio, even though, naturally, its base unit is ‘one’.

According to Dybkaer’s proposal, number-of-entities, radian, and molecular weight have the same base unit and the same base dimension. Isn’t this odd? No, not if two facts about the SI are kept clearly in mind. First, the system contains a distinction between quantities of the *same dimension* and quantities of the *same kind*; and the expression ‘number of entities (of elementary entity E)’ can and should be regarded as referring to quantities of a different kind than the quantities ‘radian’ and ‘molecular weight’ refer to. There are, by the way, reasons also to claim that radian and molecular weight are of different kinds; they have radically different origins, i.e., $(\text{length}) \cdot (\text{length})^{-1}$ and $(\text{mass}) \cdot (\text{mass})^{-1}$, respectively.

Second, the SI allows (and Dybkaer’s proposal implies) that a derived quantity can have the same dimension and base unit as a base quantity. For instance, explicitly recognized in the SI, the dimension of rainfall is: (areic volume) = (volume per area) = $(\text{length})^3 \cdot (\text{length})^{-2}$ = (length). In such a case, not only is the expression ‘length of rod X’ referring to another kind of quantity than ‘areic volume of rainfall Y’; the ordinary meaning of the term ‘length’ is gone, too, since areic volume is not something one-dimensionally extended in space. The same thing happens with ‘number of entities’ in Dybkaer’s proposal. Derived quantities such as radian and molecular weight obtain the same dimension as the *base* dimension ‘number of entities’, but this implies that the term ‘number of entities’ takes on a much wider meaning than the traditional.

There is a simple philosophical reason why the notion ‘amount of substance’ cannot be used before it is specified: one kind of substance can contain other kinds of substances as parts. One mole of molecule O_2 contains the same amount of substance as 2 mole of atom O, i.e., depending on whether the substance is measured in molecules or atoms, one obtains the quantity ‘2 mole’ or the quantity ‘1 mole’. This observation is of course equally relevant for the dimension ‘number of entities’. Dybkaer’s proposal for the base unit one, accordingly, contains a qualification too. The proposal is originally put forward in a form that fits VIM (Dybkaer, 2004a, p. 72), but I have made it as similar to the SI introduction of the mole as possible (also, I have left out of account a special name proposal of Dybkaer and Mills). My re-formulated Dybkaer proposal looks like this:

> 1. The unit one is the base unit for number of entities corresponding to a single countable defined entity; its symbol is “1,” but this symbol may be deleted.
> 2. When the one is used as a base unit, the type of object must be defined and may be, e.g., a particle, system, phenomenon, or event.

This means that quantity expressions using this base unit have a four-partite structure just like the one for the un-abbreviated mole expressions:

- QS = x · one · (number of entities) · (of elementary entity E)
- QS = x · mole · (amount of substance) · (of elementary entity E)

Let us now take a new look at the semantic holism in the introduction of the mole. If Dybkaer’s proposal had been fully accepted, one could have said that the introduction of the base unit mole presupposes an understanding of the (eighth) base dimension, ‘number of entities’ and its base unit ‘one’. This dependence would then be on an equal footing with the dependence that exists in the introduction of the meter; where meter is made dependent on the base dimension time and the base unit second. However, there is in the ‘mole-to-one’ case more things to be said than in the ‘meter-to-second’ case; things that Dybkaer seems not to have noticed.

The number of atoms in 0.012 kilogram of carbon 12 is a well known number, $6.02 \cdot 10^{23}$, with a famous name, ‘the Avogadro number’. As made explicit in the SI brochure (SI 2006, p. 115), what is said in point 1 of the introduction of the mole can also be stated as a general relationship between number of entities and mole amounts; it is reminiscent of Avogadro’s gas law. Let (i) N be the quantity value of the number of entities of kind E in a sample X, (ii) n be the quantity value of the amount of substance of this kind in the same sample, and (iii) A be the Avogadro number; then it holds true:

- o $N(\mathrm{X}) = \mathrm{A\ mole}^{-1} \cdot n(\mathrm{X})\ \mathrm{mole} = n(\mathrm{X})\ \mathrm{mole} / \mathrm{A\ mole}$

Since ‘N(X)’ is short for ‘N(X) one’, the formula above conforms to the purely dimensional formula below:

- (number of entities) = (amount of substance)$^{-1}$ · (amount of substance)

In the first formula, the purely mathematical Avogadro number is conjoined with the base unit ‘mole^{-1}’ and turned into a quantity, called ‘the Avogadro constant’. Were it not for the fact that the expression ‘A mole’ does not refer to an amount of substance of any specific kind, but merely to a physical constant, then the dimension ‘number of entities’ could (contrary to Dybkaer’s proposal) be regarded as a derived quantity dimension. Since it cannot be so derived, there is, as made clear in both the SI and VIM, a real need for a base dimension ‘number of entities’. However, there is also another dimensional way to state the Avogadro relationship. One can refuse to combine the Avogadro number with a base unit. Such a move makes the Avogadro constant identical to the Avogadro number, i.e., *the constant becomes a pure mathematical number instead of a quantity*. One then obtains:

- N(X) = A · n(X) mole
- N (number of entities) = (mathematical number) · n (amount of substance)

Since A is not here conjoined with a base unit, the equality statement ‘N(X) = A · n(X) mole’ is of exactly the same kind as ‘N meter = 3.28 · n foot’, i.e., it belongs to the kind of statements that tell how two different base units for the same dimension are related to each other. In this way, Dybkaer’s proposal can be turned into a proposal that makes the terms ‘number of entities’ and ‘amount of substance’ into two terms that bring in two different base units but name the same physical dimension.

From a philosophical point of view, I can see no disadvantages with such a re-interpreted Dybkaer proposal. It does not (unlike Dybkaer’s original proposal) make ‘number of entities’ and ‘one’ into the dimension and unit of a new eighth base quantity; it wants them substituted for ‘amount of substance’ and ‘mole’. From a metrological point of view, such an exchange may be confronted by pragmatic obstacles that I cannot see, but, for sure, the proposal does not imply a ban on the use of the mole unit. Since the SI explicitly allows several ‘non-SI units’ to be used together with the SI units (SI 2006, Chapter 4), ‘mole’ should of course then be regarded as belonging to this group, which otherwise contains units such as ‘minute’, ‘degree’, and ‘litre’.

We are now in a position where light easily can be shed on the two misunderstandings of the mole mentioned. According to the first one, the mole is simply the purely mathematical number $6.02 \cdot 10^{23}$, the number of atoms in 1 mole of carbon 12. At present (February 2008), it occupies Wikipedia's entry for 'Mole (unit)', which says: 'A mole is much like "a dozen" in that both are absolute numbers (having no units) and can describe any type of elementary object.' The second one can be found in the computer dictionary 'Whatis.com', which says (February 2008): 'The mole is the only fundamental SI unit that is dimensionless', and 'Amount of substance, also called material quantity, is a dimensionless expression of the number of particles in a sample.' I take it that the expression 'is dimensionless' should here not mean 'is a mathematical number', but (as in VIM and the SI) 'has dimension one'.

The misunderstanding that the mole has dimension one can be seen as a way of spontaneously and implicitly doing what I have been at pains to work out explicitly, namely to conceive the mole as a base unit for the same dimension as normally the SI unit 'one' is a base unit for; be this dimension called 'number of entities' or 'dimension one'. The Wikipedia view that the mole is only a mathematical number can then be looked upon as first performing the move just presented, and then conflating the SI unit one with the mathematical number one. In my opinion, one should not be too harsh toward the Wikipedia author. Quite an effort is needed to keep the SI unit one and the mathematical number one apart; especially since even in VIM (see quotation above) the base unit for 'number of entities' is called 'the *number* one' not 'the (metrological) *unit* one'.

5. Functioning as a purely qualitative physical dimension with degrees

In the preceding sections, I have distinguished between two-partite quality expressions (QL), three-partite quantity expressions (Q), and four-partite quantity expressions with specification (QS). Their abstract forms, to repeat, are:

- QL = (quality value) · (physical dimension)
- Q = (quantity value) · (base unit) · (physical dimension)
- QS = (quantity value) · (base unit) · (physical dimension) · (specification)

In this section, I will present and discuss a fourth such form, the form for *functionality expressions* (F*); a kind of expressions that must not be conflated with *function* expressions (F). The latter can be found in the classic examples of the philosophy of teleology: 'the *function* of chlorophyll in plants is to enable photosynthesis' and 'the function of the heart is to pump blood'; whereas functionality expressions can be found in the following statements: 'the *functioning* of the heart is perfect', 'the functioning of the heart could be better', 'the functioning of the heart is really bad', and 'the heart has stopped functioning'. As I will use this section to make clear: on the same level of abstractness and explicitness as the three Q-forms above, functionality expressions (F*) take on this four-partite form:

- F* = (functionality value) · (prototype unit) · (physical dimension) · (specification)

On the surface, the structure of F* is similar to that of QS, but there is a gulf between them. First, F* has nothing to do with the dimension 'amount of substance' or 'number of entities'. Second, each functioning (F*) is a functioning of a function (F), and function is not a quantifiable dimension (see Section 2). This is also the reason why the notion of 'base unit' has to be exchanged for 'prototype unit'. As we will see, a natural name of the F*-dimension is 'degree of functioning', even though this name may give rise to unwanted quantity associations; the natural prototype unit is 'proper functioning' (or '100% functioning'). Inserting these notions into the F*-form above, we obtain:

- F* = (functionality value) · (proper functioning) · (degree of functioning) · (specified function)

Functionality statements do, like amount-of-substance statements, easily lend themselves to abbreviations. A statement such as 'his heart is not functioning properly' can be spelled out as 'his heart, as part of his human organism, is not functioning properly'; among medical people it might be shorthand for 'his heart, as part of the human circulatory system, is not functioning properly'. If we allow 'not functioning properly' to be synonymous with 'is only functioning to 80%', the last statement can be given the two following more concrete forms (which in due course will be explained):

- F* = (not) · (proper functioning) · (degree of functioning) · (function of heart in the human circulatory system)
- F* = (0.8) · (100%) · (degree of functioning) · (pumping blood in human)

The main thesis of this section can be stated thus:

- *functionality expressions do not suffer from any metrological defects; they are from a metrological-scientific point of view as respectable as quantity expressions are*.

On this view, the four expressions ‘helix shape’, ‘5.03 m long’, ‘1.42 kg mass’, ‘1 mole of HgCl’, and ‘not functioning properly’ are equally metrologically mature. Let me now argue for the claims made about functionality expressions.

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To start with, I have to divert the view that to ascribe functions to something in nature is no more than to project goals and purposes onto nature, i.e., divert the view that function ascriptions are only anthropomorphizations and, therefore, nothing that science can take seriously. Function talk seems to come as naturally in biology, medicine, and molecular biology as talk about shapes, lengths, and weights come naturally in physics and chemistry. Nonetheless, the Darwinian revolution has made literal function ascriptions fall into disrepute in mainstream philosophy of science. It is here taken to be a way of retaining the very concepts of ‘God’s design’ and ‘Nature’s internal purpose’ that Darwins’s theory exchanged for ‘natural selection’. According to this view, only when understood as ‘Man’s metaphorical ascription’, should natural scientists be allowed to say that a thing, system, property, or process exists *for the sake of* something, or has something as a *goal*. That is, statements such as ‘the heart exists for the sake of pumping blood’ and ‘the goal of the heart activity is to circulate blood’ can never be literal descriptions. Admittedly, from a natural-scientific causal point of view, goals have a strange feature. The goal of a present activity can be situated in a distant future, whereas the effects of ordinary causes have to be either simultaneous with the cause or appear in the very next moment (Johansson 2006a).

Nonetheless, there are nowadays quite a number of critics who argue that the concept of ‘function’, as used in the contemporary life sciences, should from a philosophical point of view be understood as having a place besides or in-between the concepts of ‘design/purpose/goal’ and ‘natural-selection-causality’; see e.g., (Cummins 1975), (Boorse 1976), (Mahner and Bunge 1997, Chapter 4.6), (Manning 1997), (McLaughlin 2001), (Ariew et al 2002), and (Krohs 2007). In its general outlines, this critical stance is in what follows taken for granted, but I need to sketch my own specific position on functions, functionings, and the identity of functional wholes (Johansson 2004, 2005, 2006a, 2006b).

In order to easily keep my view on functions distinct not only from the concepts of ‘designed functions’, ‘internal purpose/goal functions’, and ‘metaphorical functions’, but also from the concepts created by the other critics, I will use two special terms; I will interchangeably talk of ‘part-to-whole functions’ and ‘constituent functions’.

A functional whole such as the human body has many functional parts, the circulatory system being one of them. If, next, the circulatory system is regarded as a functional whole, then it can be ascribed functional parts, the heart being one such. If, in turn, the heart is regarded as a functional whole, one has to discern its functional parts (atria, ventricles, and valves). And so on. This means that the correlative notions of ‘functional whole’ and ‘functional part’ can be parts of hierarchical structures of the following kind:

$$\begin{tikzcd}
\text{functional whole}_0 \arrow[d, shift right] & & \\
\text{functional part}_1 \arrow[u, shift right] \arrow[r, phantom, "="] & \text{functional whole}_1 \arrow[d, shift right] & \\
& \text{functional part}_2 \arrow[u, shift right] \arrow[r, phantom, "="] & \text{functional whole}_2
\end{tikzcd}$$

My claim is that the function of a functional part can (independently of whether Darwin is right or wrong) be scientifically studied *in its relation to* a functional whole *that is already ascribed a function* of some sort; the function of the functional part is then studied as a *constituent function*. The point of such studies is, among other things, to distinguish between those causal effects of a part that exist for the sake of the functional whole and those that do not. To take the classic example in this respect: the function of the heart is to pump blood, not to produce heart sounds, which it also does (Hempel 1965, p. 305). The same observation applies to designed artifacts. For instance, in times of war, engineers can be delivered an unknown kind of technological enemy device (aircraft, missile, etc.), and be ordered to find out how it *functions*. With respect to the parts of the device, they then have to distinguish between causal effects that are accidental and causal effects that are essential to the purpose of the device, i.e., are functional.

Let us assume that the functional whole$_0$ in the function hierarchy above has no functional whole above it. This means that the functions of part$_1$/whole$_1$ and part$_2$/whole$_2$ can be studied as constituent functions, but that the function of the functional whole$_0$ cannot. Its function has to be regarded (i) as being designed by a deity, or (ii) as being an intrinsic purpose implanted by nature, or (iii) as being a consciously metaphorical ascription of a purpose to a purely causal system (e.g., to a system that has emerged thanks to natural selection). If the functional whole$_0$ is the human organism, and it is assumed that its short-term function is selfreproduction, i.e., ‘to preserve its own life’, then one can start a philosophical discussion about whether: (i) some higher being has designed our body with the life-preserving purpose in mind, (ii) the body contains by its own nature a life-preserving purpose, or (iii) we merely project such a life-preserving purpose onto the body because we would like the world to be this way. Empirical investigations that would settle this issue seem not to be in sight.

The same three interpretative possibilities are present in relation to the functions of whole$_1$ and whole$_2$, but in these cases there is also a fourth possibility. One can completely neglect the alternatives that their function is (i) God-designed, (ii) Nature-given, or (iii) Man-projected, and instead ask what (iv) Constituent function the functional part has *in relation to* the *pre-given function* of the whole above it. And now empirical investigations become highly relevant. On the *assumption* that the short-term function of the human body is to preserve its life, then it is no longer a philosophical but an empirical question what constituent function the circulatory system has. Similarly, if the circulatory system (part$_1$/whole$_1$) is already ascribed the function ‘to transport substances between bodily systems’ (be it by God, Nature, or Man; or be it a Constituent function), then it is an empirical question what constituent function the heart (part$_2$/whole$_2$) has in relation to the circulatory system.

Let me put this observation in a wider framework. Think of game theory. It is devoted to the study of means-*to*-end rationality. Given certain preferences and a certain situation, it studies what are the most rational means to satisfy the preferences. The results reached are just as objective as other scientific results are; and this independently of whether the preference systems in question can be regarded as rational or not. Put in other words: game theoretical scientific investigations of rationality can depart from preference systems that are completely subjective and irrational. Now, as game theory can objectively study ‘means-*to*-end rationality’ independently of the possible subjectivity and irrationality of the posited end, I claim that the life sciences can objectively study ‘part-*to*-whole functionality’ (constituent functions) independently of the possible subjectivity and purposelessness of the function ascribed to the whole. That is:

- *a Constituent function can be objectively studied independently of the ontological status of the function of the encompassing functional whole.*

The analogy made is not accidental. In both cases there is teleology. Both the posited action end and the function of the whole are in some sense goals. The remarks made show: that even if questions about divine design and purposes-in-themselves – final-end-teleology –

are beyond the reach of science, questions about means-teleology are not. Therefore, it is wrong to ban completely from science the notion ‘for the sake of’. A constituent function is a function that exists for the sake of a function of a functional whole; nonetheless it can be empirically studied.

According to the analysis made, there are four ways of answering the question why a causal system S is a functional whole:

(i) God has designed S to be a functional whole
(ii) Nature contains in and of itself functional wholes of kind S
(iii) Man has chosen to metaphorically regard S as if it were a functional whole
(iv) S is merely a constituent functional whole.

In the last case, one of the first three answers has from a philosophical point of view to be brought in later in order to stop the regress that constituent functions bring with them, but from an empirical-scientific point of view this absolute end point can be left out of account.

The notion of ‘constituent function’ being accepted, one can ask what constituent function ascriptions should look like in detail. Since a functional whole contains and supervenes on more than one part, and the parts have to interact causally, a characterization of the constituent function (F) of a certain functional part (B) has to mention both the functional whole (A) and something (X) in relation to which B produces some effects. Here is a proposal (Johansson, Smith, et al 2005, p. 159-62):

- in the functional whole A, one constituent function of the functional subunit B of A is: to F in relation to X, Y, Z, etc.

Three examples:

- in the human body (A), one constituent function of the circulatory system (B) is: to transport substances (F) between the bodily systems (X, Y, Z, etc.)
- in the circulatory system (A), one constituent function of the heart (B) is: to pump blood (F) in the blood vessel system (X)
- in the alimentary system (A), one constituent function of the liver (B) is: to produce glycogen (F) that is necessary for energy storage (X).

Note that this form deletes the ancient view that an entity can have only one function. The liver, for instance, has not only the function to produce glycogen; it should also produce bile and perform some further functions as well. Also, note that the expression ‘the functional subunit B of A’ allows that completely different kinds of structures (B) can perform the same constituent function (F); both a normal human heart and an artificially (not yet!) created heart made of metal, plastic, and chips can pump blood. Multi-functionality and prosthetic devices are today obvious facts.

This characterization of constituent functions turns a common definition pattern from chemistry and physics upside down. Matter unities (wholes) are often regarded as being defined by their material parts; a water molecule, for example, should by definition consist of two hydrogen atoms and one oxygen atom. But a constituent function of a functional part becomes defined by the whole of which it is a part; the function of the part B is defined as ‘F--in-relation-to-A’. This difference can briefly be stated thus: in the case of purely material unities, a whole can be defined by its parts, in the case of constituent functions, a part is defined by its whole.

Constituent functions can be objectively and scientifically studied without necessarily (but, I would say, preferably) being put in a Darwinian framework. From a metrological point of view, to accept the notion of ‘constituent function’ means only to accept that function talk can be on a par with shape talk. ‘Constituent function’ can be taken to refer to a physical dimension that has many different determinate constituent functions as its quality values. But there is more to be said from a metrological perspective. Let me proceed, but in what follows with the term ‘constituent function’ mostly abbreviated into just ‘function’.

Necessarily, where there are functions there can be functionings. A screwdriver has always the designed function to drive screws, even if it is only now and then in functioning, i.e., actually used to drive a screw. A human organ such as the heart, however, not only always has the constituent function to pump blood; it is always in a state of functioning. Now, as the notion of ‘length’ comes with the idea that there can be different determinate lengths, and ‘substance’ comes with the idea that there can be different determinate amounts of a certain kind of substance, ‘function’ comes with the idea that a function can, when functioning, be performing its function more or less well. A function is a feature (property or disposition) that something either has or has not; a function does not take degrees. But the corresponding states of functionings do.

Let me repeat: *functionings take degrees*. Therefore, with respect to the functioning of constituent functions, it makes good sense to regard degree of functioning as a physical dimension whose determinate functioning-values are open to estimation. From an everyday practical perspective, it is often of interest to distinguish between at least four broad value ranges: proper functioning, acceptable functioning, malfunctioning, and not functioning. A functional part such as the liver, which has more than one function, can be functioning acceptably with respect to one constituent function (e.g., glycogen production) but be malfunctioning with respect to another (e.g., bile production).

Instead of saying ‘it is properly functioning’, people quite often say ‘it is 100% functioning’. Sometimes also the sentence ‘it is acceptably functioning’ is exchanged for sentences such as ‘it is 80% functioning’ or ‘it is 70% functioning’. Having stopped functioning altogether is then of course the same as ‘0% functioning’. This use of numerals does not imply that functionings can be linearly ordered the way the SI dimensions have been given a strict logico-mathematical such ordering. It means that numerals can be put to other uses than referring to numbers. The degrees and percentage numbers spoken of in relation to functionings are of a very informal kind. Therefore, it is impossible to introduce a base unit for degree of functioning; but there is nonetheless something that plays the role of a base unit. It is called ‘prototype’. Let me explain.

In everyday language, the sentence form ‘he/she/it is a *typical* X’ is now and then used. Both cognitive scientists and linguists have empirically studied what most people take to be such typical Xs; this has been done in relation both to terms for artifacts such as ‘furniture’ and species terms such as ‘bird’. Our perceptual apparatus seems to have an in-built capacity to identify something as being typical of its kind, and then relate other, both typical and non-typical, things to this prototype. When children learn a language, they first learn to connect words to prototypical instances. As Eleanor Rosch has stressed, it is important even for researchers to realize that there are two different kinds of classification: logical and prototypical, respectively (Rosch 1983). In this terminology, all the SI quantities are logical classifications, whereas degree of functioning is a prototypical classification. *The idea of a perfect functioning of a certain kind is an idea of a prototype*. It is no base unit, but it plays the role of a base unit in the sense that it is a point of reference in relation to which instances of states of functioning can be given an informal distance measure (cf. the end of Section 2 above and Johansson (2004)). Mostly, trained experts are needed in order to estimate such informal distances (e.g., physicians in relation to heart functioning).

If these views on functionings are to be put in a table like Tables 1 and 4, we obtain Table 6.

Base Pure-Quality	***Prototypical Unit***	***Symbol***
degree of functioning	proper functioning	100%

Table 6: *Degree of functioning as a pure quality dimension with a prototypical ('base') unit.*

From Table 6 it follows that an F*-expression must have at least three parts: '(functionality value) · (proper functioning) · (degree of functioning)'. I have, however, claimed that such expressions consist of four parts; 'specified function' is still missing. This is as it should be. Not even in the case of a QS-expression ('amount of substance') is the four-partite structure derivable from what can (correspondingly) be seen in Table 1. Further considerations were needed, and the same applies for F*-expressions. But, as with amount of substance, it can easily be shown that degree of functioning requires specification. It follows from the existence of prosthetic devices and xenotransplantations.

In order to be able to stay with the heart as our prime example, let us assume that pig hearts can be made to function as substitutes for human hearts. Such a pig heart is when working in a human body performing the same constituent function as the original human heart was. Using the formula for constitutive functions we can say: 'in the human circulatory system (A), the first-pig-and-then-human heart (B) performs the constituent function of pumping blood (F)'. Necessarily, a constituent function F is an F-in-relation-to-A. This means that the pig heart has one constituent function in the pig and another constituent function in the human body; constituent functions are part-to-whole functions that receive part of their identity from the whole in which they are functioning.

Now, I take it, a heart activity that is not a proper functioning in the human body, might well be a proper functioning in the original pig organism. The point made becomes even clearer if we assume the existence of artificial hearts that can pump blood in humans, but pump water in some hydrological devices; then, what is a proper functioning in the human body might obviously not be so in all the hydrological devices. Therefore, *a prototypical functioning is a prototype only in relation to a constituent function of a given functional whole*; different such functions are in Table 2 listed by means of the symbols F_1 to F_n.

Assume, as above, that a pig heart that is properly (100%) functioning within a pig is transplanted into a human, but that in this new body the heart is only 80% functioning. In order to describe these two situations the following functionality expressions could be used:

- F_1* = (1) · (100%) · (degree of functioning) · (pumping blood in pig)
- F_2* = (0.8) · (100%) · (degree of functioning) · (pumping blood in human).

The same kind of activity that in F_1* is given the functionality value 1, is in F_2* given the value 0.8. But this is nothing odd, since the F_1*- and F_2*-expressions differ in their specification parts. This shows that, when everything is made explicit, *functionality expressions have to have a four-partite structure*.

When numerals are used as functionality values, they do not refer to mathematical numbers, but express informal estimations made by experts. There might, though, be cases where some range of functionality values can, in a law-like way, be connected to a variable for an already quantified physical dimension. The functioning of the heart may perhaps, with some other factors held constant, be linked to temperature degrees. Such a restricted quantification would be to degree of functioning what the quantifiability of ellipses is to the non-quantifiability of shapes.

Just as the liver can have two functions, prototypes can. In the history of biology and medicine, prototypes have not only been given the metrologically normative function laid bare above; they have often also been given an ethically normative function (Johansson 2004, Section 7). Before Darwin, most European taxonomists thought that God had designed all the species, and that, therefore, non-prototypical exemplars of species were in some sense ethically imperfect, and not 'as they should be'. This old fusion of biological-metrological and biological-ethical normativity is for many people still present in the combined views that what is prototypical must be natural, and what is natural must be ethically correct. In medicine, there is something similar. Metrological prototypes of functionings are often fused with what is regarded as non-diseased functionings. It would be odd, but not logically impossible, to take sick humans as metrological prototypes for some functionings; but this would mean that 'proper (100%) functioning' could no longer be chosen as a prototypical base unit, since then there would be functionality values both above and below the prototype, and nothing can be 'above' proper functioning.

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Let me summarize the consecutive steps of the argument of this section:

(a) there is one kind of functions, constituent functions, that can be objectively and empirically-scientifically studied
(b) a function is either at rest or in a state of functioning
(c) such functionings take, in a very informal sense, degrees
(d) such informal degrees are estimated against a prototype (prototypical base unit) that represents proper functioning

(e) a functionality expression for the functioning of a constituent function has the following four-partite abstract form: F* = (functionality value) · (proper functioning) · (degree of functioning) · (specified function)

(f) the unit ‘proper functioning’ is from a scientific point of view only a metrological norm.

And the over-arching conclusion is:

(g) the four-partite form for functionality expressions is as metrologically good as the four-partite form for the SI dimension ‘amount of substance’.

6. The book of nature is not wholly written in the language of mathematics

Leaving luminous intensity aside, the following story about the SI base dimensions can be told. (Luminous intensity is used only in photometry, which is concerned with physiological responses to light, and has become somewhat obsolete; the corresponding dimension in relation to electromagnetic radiation is ‘radiant intensity’, which is a derived dimension.) Classical mechanics uses only the first three base dimensions, i.e., length, mass, and time; the classical electromagnetic theories require a fourth base dimension, and electric current has been chosen; thermodynamics requires temperature; and physical chemistry amount of substance. However, neither classical mechanics nor physical chemistry can do wholly without the physical dimension of shape. In classical mechanics one wants to talk about the shapes of the movements of projectiles and heavenly bodies, and in physical chemistry one wants to talk about the shape of molecules. Since, as I have shown, shape is necessarily a pure quality dimension, the book of nature is not wholly written in the language of mathematics.

The scientific development since the 1950s has abolished the former research gap between chemistry, on the one hand, and biology and medical science, on the other. As a manifestation of this change, the umbrella term ‘the life sciences’ has emerged and become popular. Function talk is no longer restricted to biology and the medical sciences; it has become firmly entrenched in molecular biology and chemistry, too. Since degree of functioning, like shape, is a purely physical dimension, this points towards the view that not even the largest part of nature is written in the language of mathematics.

This being said and claimed, I want now to stress what I have earlier said only in passing: a pure quality dimension can allow that one or several of its subsets of property determinates become quantified. If such a partial quantification fulfills a scientific or practical purpose, then it should of course be made. Let me make an analogy. One should not try to build a perpetual motion machine, i.e., a machine that produces more energy than it uses, since such a machine is theoretically impossible, but one should nonetheless try to build as energy effective machines as possible. Similarly, one should not try to quantify shapes and functions-functionings, since this is theoretically impossible, but one should nonetheless quantify subsets when this is both possible and useful.

The main message of this paper can be put in a single sentence: the book of nature is partly written in the language of mathematics, partly in the language of functions and partly in a very simple purely qualitative language.

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